Received: 23 April 2020

DOI: 10.1002/mma.7261

RESEARCH ARTICLE

WILEY

An accelerated hybrid projection method with a self-adaptive step-size sequence for solving split common fixed point problems

Zheng Zhou | **Bing Tan** | **Songxiao Li**

Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China

Correspondence
Zheng Zhou, Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China,
Email: zhouzheng2272@163.com

Communicated by: H. M. Srivastava

This paper attempts to solve the split common fixed point problem for demicontractive mappings. We give an accelerated hybrid projection algorithm that combines the hybrid projection method and the inertial technique. The strong convergence theorem of this algorithm is obtained under mild conditions by a self-adaptive step-size sequence, which does not need prior knowledge of the operator norm. Some numerical experiments in infinite-dimensional Hilbert spaces are provided to illustrate the reliability and robustness of the algorithm and also to compare it with existing ones.

KEYWORDS

demicontractive mapping, hybrid projection method, inertial technique, self-adaptive step-size sequence, split common fixed point problem

MSC CLASSIFICATION

47H09; 47H10; 47J25; 65K15

1 | INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$ and $F(K)$ be the fixed point set of a mapping K. Recall that the mapping $K : \mathcal{H} \to \mathcal{H}$ is said to be a strictly pseudocontractive if there exists a constant $\eta \in [0, 1)$ such that

$$\|Kz - Ky\|^2 \le \|z - y\|^2 + \eta\|z - Kz - (y - Ky)\|^2, \forall z, y \in \mathcal{H}$$

and is said to be a demicontractive if $F(K) \neq \emptyset$, there exists a constant $\eta \in (-\infty, 1)$ such that

$$\|Kz - p\|^2 \le \|z - p\|^2 + \eta\|Kz - z\|^2, \forall p \in F(K), z \in \mathcal{H},$$

or equivalently

$$\langle z - p, z - Kz\rangle \ge \frac{1-\eta}{2}\|z - Kz\|^2, \forall p \in F(K), z \in \mathcal{H}.$$

Both the demicontractive mapping and the strictly pseudocontractive mapping were studied by many authors.[1-4] Besides, Suparatulatorn et al.[5] and Yao et al.[6] considered the other case, which is the demicontractive mapping with coefficient $\eta \in [0, 1)$. Obviously, this case is contained in the demicontractive mapping with coefficient $\eta \in (-\infty, 1)$ that we are going to study. Meanwhile, the strictly pseudocontractive mapping with $F(K) \neq \emptyset$ is the demicontractive mapping. The opposite is contradictory. Further, according to the properties of the norm and the inner product in Hilbert spaces, we know that the fixed point set of a demicontractive mapping is closed and convex.

In addition, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces, C and Q be two nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator. The split feasibility problem (Censor and Elfving[7] introduced in 1994) is to find $z^* \in C$ such that $Az^* \in Q$. Further, let $K : \mathcal{H}_1 \to \mathcal{H}_1$ and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings. The split common fixed point problem (Censor and Segal[8] introduced in 2009) is to find

$$z^* \in F(K) \text{ such that } Az^* \in F(S). \tag{1}$$

If $K = P_C$ and $S = P_Q$, where $P_C : \mathcal{H}_1 \to C$ and $P_Q : \mathcal{H}_2 \to Q$ are metric projection mappings, then the split common fixed point problem is equivalent to the split feasibility problem. As we all know, z is a solution of problem (1) if and only if z is a solution of the fixed point equation $z = K(I - \mu A^{\mathrm{T}}(I - S)A)z$. Furthermore, Censor and Segal[8] proposed the following algorithm to solve problem (1) for two directed operators: $z_{n+1} = K(I - \mu A^{\mathrm{T}}(I - S)A)z_n$, where μ is a positive constant and A^{T} is the matrix transposition of A. Moreover, they studied the convergence of the sequence $\{z_n\}$ generated by this algorithm.

In the sequel, the split feasibility problem and the split common fixed point problem have received widespread attention by many authors, such as Moudafi,[9,10] Takahashi and Yao,[11] Qin et al.,[12] Cui and Wang,[13] Vinh and Hoai,[14] Liu,[15] Zhou et al.,[16,17] Masad and Reich,[18] Reich and Tuyen,[19] and Reich et al.[20] It turns out that some studies only guaranteed weak convergence of the sequence. To fill this gap, some strong convergent results of problem (1) were established by employing the Halpern algorithm and the viscosity algorithm, for instance, Boikanyo,[3] Kraikaew and Saejung,[21] and Qin and Yao.[22] Recently, Wang[23] proposed the new iterative algorithm to approximate the solution of the split common fixed point problem: $z_{n+1} = z_n - \mu_n[(I-K) + A^*(I-S)A)]z_n, \ \forall n \geq 1$, where $\{\mu_n\}$ is a self-adaptive step-size sequence and A^* is the adjoint operator of A. Meanwhile, some convergence properties were obtained under some mild conditions.

In addition, in 2003, Nakajo and Takahashi[24] introduced the following hybrid projection method to study the fixed point problem and guaranteed strong convergence property of the following algorithm:

$$\begin{cases} y_n = \beta_n z_n + (1 - \beta_n)Kz_n, \\ C_n = \{u \in C : \|y_n - u\| \leq \|z_n - u\|\}, \\ Q_n = \{u \in C : \langle z_n - u, z_1 - z_n\rangle \geq 0\}, \\ z_{n+1} = P_{C_n \cap Q_n} z_1, \ n \geq 1. \end{cases} \tag{2}$$

Recently, methods based on projection types have been studied by many scholars to solve various problems, see, for example, Zhou et al.,[16] Reich and Tuyen,[19] and Tan et al.[25] and the references therein. On the other hand, in 2001, Alvarez and Attouch[26] proposed an inertial proximal algorithm to study the inclusion problem of a maximal monotone operator:

$$z_{n+1} = J^T_{\lambda_n}(z_n + \alpha_n(z_n - z_{n-1})), \ \forall n \geq 1,$$

where the set-valued mapping $T : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximal monotone operator and the function $J^T_{\lambda_n} = (I + \lambda_n T)^{-1}$ is the resolvent of T. The extrapolation term $\alpha_n(z_n - z_{n-1})$ takes into account an inertial effect of this algorithm. Under some mild conditions, most iterative algorithms by using this inertial effect have better convergence behavior for various problems, such as the inclusion problem,[27-29] the variational inequality problem,[30,31] and the fixed point problem.[32,33] Unfortunately, the selection of parameters related to the iterative algorithm usually causes interference in the process of approximate solution. Hence, the selection of parameters is also very meaningful in future research.

Based on the ideas of Wang,[23] Nakajo and Takahashi,[24] and Alvarez and Attouch,[26] we propose a new inertial hybrid projection algorithm to approximate the solution of problem (1) for demicontractive mappings by combining the inertial technique and the hybrid projection method. Simultaneously, we introduce a new self-adaptive step-size sequence, which does not need prior knowledge of the operator norm. Under mild conditions, the corresponding strong convergence theorem is obtained in infinite-dimensional Hilbert spaces. It is worth noting that such a self-adaptive step-size sequence guarantees the stability of the proposed algorithm. Finally, some numerical experiments in infinite-dimensional Hilbert spaces are used to demonstrate the efficiency of our main results.

2 | PRELIMINARIES

For the convenience in the rest of this article, let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. $\omega_w(z_n)$ denotes the set of all weak cluster points of a sequence $\{z_n\}$, and the notations $\to$ and $\rightharpoonup$ represent strong convergence and weak convergence, respectively. Let P_C denote the metric projection from $\mathcal{H}$ onto C, that is, $P_C z = argmin_{y\in C}\|z-y\|$, $\forall z\in\mathcal{H}$. The following property holds.

$$\langle P_C z - z, P_C z - y\rangle \le 0 \iff \|y-P_C z\|^2 + \|z-P_C z\|^2 \le \|z-y\|^2,\ \forall y\in C. \tag{3}$$

For these and more properties of the metric projection P_C, see, for example, Goebel and Reich[34] (chapter 3) and Kopecká and Reich.[35] And then, it is easy to know that a Hilbert space $\mathcal{H}$ has Kadec–Klee property, that is, a sequence $\{z_n\}\subset\mathcal{H}$ that satisfies $z_n \rightharpoonup z$ and $\|z_n\|\to\|z\|$, then $z_n\to z$. In Takahashi,[36] for any $z, y\in\mathcal{H}$, the following results hold:

(I) $\|z+y\|^2 = \|z\|^2+\|y\|^2+2\langle z,y\rangle \le \|z\|^2 + 2\langle y, z+y\rangle$;
(II) $\|\alpha z+(1-\alpha)y\|^2 = \alpha\|z\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|z-y\|^2$, $\forall\alpha\in\mathbb{R}$.

Definition 1. The mapping $K:\mathcal{H}\to\mathcal{H}$ and $F(K)\neq\emptyset$. $I-K$ is demiclosed at 0, if and only if, for any sequence $\{z_n\}\subset\mathcal{H}$, satisfying $z_n\rightharpoonup z$ and $(I-K)z_n\to 0$, then $z\in F(K)$.

Lemma 1 (Zhou and Qin[37] and Marino and Xu[38]). *Let C be a nonempty closed convex subset of a Hilbert space $\mathcal{H}$. Let $K:C\to\mathcal{H}$ be a strictly pseudocontractive mapping with coefficient $\eta\in[0,1)$. Then, the fixed point set $F(K)$ is closed and convex, and $I-K$ is demiclosed at 0.*

3 | MAIN RESULTS

In the following, we introduce a self-adaptive inertial hybrid projection algorithm to solve the split common fixed point problem for demecontractive mappings and prove the strong convergence property of this algorithm.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces, $A:\mathcal{H}_1\to\mathcal{H}_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $K:\mathcal{H}_1\to\mathcal{H}_1$ and $S:\mathcal{H}_2\to\mathcal{H}_2$ be demicontractive mappings with coefficients $\eta_1\in(-\infty,1)$ and $\eta_2\in(-\infty,1)$, respectively. For any initial points $z_0, z_1\in\mathcal{H}_1$, the sequence $\{z_n\}$ generated by the following iterative process.

$$\begin{cases} w_n = z_n + \vartheta_n(z_n - z_{n-1}), \\ u_n = w_n - \mu_n\left[(I-K)w_n + A^*(I-S)Aw_n\right], \\ C_n = \left\{u\in\mathcal{H}_1 : \|u_n-u\|^2 \le \|w_n-u\|^2 - \mu_n\theta_n\right\}, \\ Q_n = \{u\in\mathcal{H}_1 : \langle z_n - z_1, z_n - u\rangle \le 0\}, \\ z_{n+1} = P_{C_n\cap Q_n} z_1,\ n\ge 1, \end{cases} \tag{4}$$

where

$$\theta_n = (1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 + (1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2.$$

Theorem 1. *Assume that the solution set $\Omega=\{z^*: z^*\in F(K), Az^*\in F(S)\}\neq\emptyset$, $I-K$ and $I-S$ are demiclosed at 0. If the following conditions hold.*

(C1) The sequence $\{\vartheta_n\}$ is bounded in $(-\infty,\infty)$.
(C2) If $(I-S)Aw_n\neq 0$, the step-size $\mu_n=\sigma_n\min\left\{\frac{1-\eta_1}{2}, \frac{(1-\eta_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n\in(0,1)$. Otherwise, $\mu_n=\sigma_n(1-\eta_1)/2$.

Then, the iterative sequence $\{z_n\}$ generated by proposed algorithm (4) converges strongly to $\hat{z}=P_\Omega z_1\in\Omega$.

Proof.

Step 1. First, we show that $\Omega\subset C_n\cap Q_n$.
It is obvious that $C_n\cap Q_n$ is a closed convex set, that is, $P_{C_n\cap Q_n}z_1$ is well defined. Put any $z\in\Omega$, that is, $z\in F(K)$ and $Az\in F(S)$. From the definition of μ_n,

$$\theta_n = (1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 + (1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2 \ge 0. \tag{5}$$

By algorithm (4), we have

$$\begin{aligned}
\|u_n - z\|^2 &= \|w_n - z\|^2 - 2\mu_n\langle (I-K)w_n + A^*(I-S)Aw_n, w_n - z\rangle \\
&\quad + \mu_n^2\|(I-K)w_n + A^*(I-S)Aw_n\|^2 \\
&= \|w_n - z\|^2 - 2\mu_n\langle (I-K)w_n, w_n - z\rangle - 2\mu_n\langle (I-S)Aw_n, Aw_n - Az\rangle \\
&\quad + \mu_n^2\|(I-K)w_n + A^*(I-S)Aw_n\|^2 \\
&\le \|w_n - z\|^2 - \mu_n(1-\eta_1)\|(I-K)w_n\|^2 - \mu_n(1-\eta_2)\|(I-S)Aw_n\|^2 \\
&\quad + 2\mu_n^2\|(I-K)w_n\|^2 + 2\mu_n^2\|A^*(I-S)Aw_n\|^2 \\
&= \|w_n - z\|^2 - \mu_n(1-\eta_1-2\mu_n)\|(I-K)w_n\|^2 \\
&\quad - \mu_n\left[(1-\eta_2)\|(I-S)Aw_n\|^2 - 2\mu_n\|A^*(I-S)Aw_n\|^2\right] \\
&= \|w_n - z\|^2 - \mu_n\theta_n,\ n \ge 1.
\end{aligned} \tag{6}$$

This implies that $\Omega \subset C_n$.
On the other hand,

$$Q_1 = \{u \in \mathcal{H}_1 : \langle z_1 - z_1, z_1 - u\rangle \le 0\} = \mathcal{H}_1,$$

this means that $\Omega \subset Q_1$. Suppose $\Omega \subset Q_m$ for some $m \in \mathbb{N}$, we have $\Omega \subset C_m \cap Q_m$. Using $z_{m+1} = P_{C_m \cap Q_m} z_1$ and the projection property, we get that

$$\langle z_{m+1} - z_1, z_{m+1} - u\rangle \le 0,\ \forall u \in C_m \cap Q_m,$$

and

$$\langle z_{m+1} - z_1, z_{m+1} - z\rangle \le 0,\ \forall z \in \Omega.$$

This implies that $\Omega \subset Q_{m+1}$. By induction, we have $\Omega \subset Q_n$. Hence, $\Omega \subset C_n \cap Q_n$.

Step 2. We show that the sequence $\{z_n\}$ is bounded and $\lim_{n\to\infty}\|z_n - z_{n+1}\| = 0$.

Since the fixed point sets $F(K)$ and $F(S)$ are closed and convex, which means that Ω is a nonempty closed convex set, so there exists a point $\hat{z} = P_\Omega z_1 \in \Omega$. By virtue of $z_{n+1} = P_{C_n \cap Q_n} z_1$, $\Omega \subset C_n \cap Q_n$ and formula (3), we have

$$\|z_1 - z_{n+1}\| \le \|z_1 - \hat{z}\|.$$

Besides, we have that $\{\| z_1 - z_n \|\}$ is bounded, that is, the sequence $\{z_n\}$ is bounded. Using the definition of Q_n and $z_{n+1} = P_{C_n \cap Q_n} z_1 \in Q_n$, we know that

$$z_n = P_{Q_n} z_1 \text{ and } \|z_1 - z_n\| \le \|z_1 - z_{n+1}\|,$$

which implies that $\{\| z_1 - z_n \|\}$ is bounded and nondecreasing. Furthermore, $\lim_{n\to\infty}\|z_1 - z_n\|$ exists. In addition, it follows from (3) that

$$\|z_n - z_{n+1}\|^2 \le \|z_1 - z_{n+1}\|^2 - \|z_1 - z_n\|^2.$$

Thus, we have $\lim_{n\to\infty}\|z_n - z_{n+1}\| = 0$.

Step 3. We show that the sequence $\{z_n\}$ converges strongly to $\hat{z} = P_\Omega z_1$.

From algorithm (4) and condition (C1),

$$\|w_n - z_n\| = \vartheta_n\|z_n - z_{n-1}\| \to 0 \text{ as } n \to \infty.$$

By the boundedness of $\{z_n\}$, there exists a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ such that $z_{n_j} \rightharpoonup p$, for any $p \in \omega_w(z_n)$. This implies that $w_{n_j} \rightharpoonup p$. In addition, A is a bounded linear operator, we have $Aw_{n_j} \rightharpoonup Ap$. Next,

$$\|u_n - z_{n+1}\|^2 \le \|w_n - z_{n+1}\|^2 - \mu_n\theta_n \le \|w_n - z_{n+1}\|^2,$$

and

$$\begin{aligned}\|u_n - z_n\| &\leq \|u_n - z_{n+1}\| + \|z_{n+1} - z_n\| \\ &\leq \|w_n - z_{n+1}\| + \|z_{n+1} - z_n\| \\ &\leq \|w_n - z_n\| + 2\|z_{n+1} - z_n\| \to 0, \text{ as } n \to \infty.\end{aligned}$$

These imply that $\{w_n\}$ and $\{u_n\}$ are bounded. Besides, it follows from formulas (5) and (6) that

$$\begin{aligned}\mu_n\theta_n &\leq \|w_n - z\|^2 - \|u_n - z\|^2 \\ &\leq (\|w_n - z\| - \|u_n - z\|)(\|w_n - z\| + \|u_n - z\|) \\ &\leq \|w_n - u_n\|(\|w_n - z\| + \|u_n - z\|) \\ &\leq (\|w_n - z_n\| + \|z_n - u_n\|)(\|w_n - z\| + \|u_n - z\|) \to 0,\ n \to \infty.\end{aligned}$$

When $(I - S)Aw_n \neq 0$, it follows from the definition of θ_n that

$$\lim_{n\to\infty} \|(I - K)w_n\| = \lim_{n\to\infty} \|(I - S)Aw_n\| = 0.$$

Since $I - K$ and $I - S$ are demiclosed at 0, we have that $p \in F(K)$ and $Ap \in F(S)$, i.e., $p \in \Omega$. From $z_n = P_{Q_n}z_1$, $\hat{z} = P_\Omega z_1$ and the weak lower semicontinuity of the norm, we obtain

$$\|\hat{z} - z_1\| \leq \|p - z_1\| \leq \liminf_{j\to\infty} \|z_{n_j} - z_1\| \leq \limsup_{j\to\infty} \|z_{n_j} - z_1\| \leq \|\hat{z} - z_1\|,\ \forall p \in \omega_w(z_n),$$

which implies that $\lim_{j\to\infty} \|z_{n_j} - z_1\| = \|\hat{z} - z_1\|$ and $p = \hat{z}$. By means of the Kadec–Klee property of Hilbert spaces, we have that $\{z_{n_j}\}$ converges strongly to $\hat{z}$. Thus, the iterative sequence $\{z_n\}$ converges strongly to $\hat{z} = P_\Omega z_1$. □

In addition, if K and S are two strictly pseudocontractive mappings with $F(K) \neq \emptyset$ and $F(S) \neq \emptyset$, respectively, we have that K and S are demicontractive mappings with $\eta_1 \in [0, 1)$ and $\eta_2 \in [0, 1)$, respectively. Further, it follows from Lemma 1 that the fixed point sets $F(K)$ and $F(S)$ are closed and convex, and $I - K$ and $I - S$ are demiclosed at 0. Therefore, we have the following corollary.

Corollary 1. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $K : \mathcal{H}_1 \to \mathcal{H}_1$ and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be strictly pseudocontractive mappings with coefficients $\eta_1 \in [0, 1)$ and $\eta_2 \in [0, 1)$, respectively. Assume that $F(K) \neq \emptyset$ and $F(S) \neq \emptyset$, for any initial points $z_0, z_1 \in \mathcal{H}_1$, the iterative sequence $\{z_n\}$ is generated by the following algorithm.*

$$\begin{cases} w_n = z_n + \vartheta_n(z_n - z_{n-1}), \\ u_n = w_n - \mu_n\left[(I - K)w_n + A^*(I - S)Aw_n\right], \\ C_n = \left\{u \in \mathcal{H}_1 : \|u_n - u\|^2 \leq \|w_n - u\|^2 - \mu_n\theta_n\right\}, \\ Q_n = \{u \in \mathcal{H}_1 : \langle z_n - z_1, z_n - u\rangle \leq 0\}, \\ z_{n+1} = P_{C_n\cap Q_n}z_1,\ n \geq 1, \end{cases} \tag{7}$$

where

$$\theta_n = (1 - \eta_1 - 2\mu_n)\|(I - K)w_n\|^2 + (1 - \eta_2)\|(I - S)Aw_n\|^2 - 2\mu_n\|A^*(I - S)Aw_n\|^2.$$

If $(I - S)Aw_n \neq 0$, the step-size $\mu_n = \sigma_n \min\left\{\frac{1-\eta_1}{2}, \frac{(1-\eta_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$. Otherwise, $\mu_n = \sigma_n(1 - \eta_1)/2$. The sequence $\{\vartheta_n\}$ is bounded in $(-\infty, \infty)$. Suppose that the solution set $\Omega = \{z^* : z^* \in F(K), Az^* \in F(S)\} \neq \emptyset$, then the iterative sequence $\{z_n\}$ generated by algorithm (7) converges strongly to $\hat{z} = P_\Omega z_1 \in \Omega$.

In particular, when the parameter $\{\vartheta_n\}$ is always equal to 0, we have the following corollary.

Corollary 2. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $K : \mathcal{H}_1 \to \mathcal{H}_1$ and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be demicontractive mappings with coefficients $\eta_1 \in (-\infty, 1)$*

and $\eta_2 \in (-\infty, 1)$, respectively. Assume that $F(K) \neq \emptyset$ and $F(S) \neq \emptyset$, for any initial points $z_1 \in \mathcal{H}_1$, the iterative sequence $\{z_n\}$ is generated by the following algorithm.

$$\begin{cases} u_n = z_n - \mu_n \left[(I-K)z_n + A^*(I-S)Az_n\right], \\ C_n = \left\{u \in \mathcal{H}_1 : \|u_n - u\|^2 \leq \|z_n - u\|^2 - \mu_n\theta_n\right\}, \\ Q_n = \{u \in \mathcal{H}_1 : \langle z_n - z_1, z_n - u\rangle \leq 0\}, \\ z_{n+1} = P_{C_n \cap Q_n} z_1, \ n \geq 1, \end{cases} \tag{8}$$

where

$$\theta_n = (1 - \eta_1 - 2\mu_n)\|(I-K)z_n\|^2 + (1-\eta_2)\|(I-S)Az_n\|^2 - 2\mu_n\|A^*(I-S)Az_n\|^2.$$

If $(I-S)Az_n \neq 0$, the step-size $\mu_n = \sigma_n \min\left\{\frac{1-\eta_1}{2}, \frac{(1-\eta_2)\|(I-S)Az_n\|^2}{2\|A^*(I-S)Az_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$. Otherwise, $\mu_n = \sigma_n(1-\eta_1)/2$. Suppose that the solution set $\Omega = \{z^* : z^* \in F(K), Az^* \in F(S)\} \neq \emptyset$, $I-K$ and $I-S$ are demiclosed at 0. Then, the iterative sequence $\{z_n\}$ generated by algorithm (8) converges strongly to $\hat{z} = P_\Omega z_1 \in \Omega$.

4 | NUMERICAL EXAMPLES

In this section, all codes were written in MATLAB R2018b and ran on a Lenovo Ideapad 720S with 1.6-GHz Intel Core i5 processor and 8 GB of RAM. First, some numerical examples in infinite-dimensional Hilbert spaces are proposed to demonstrate the effectiveness and realization of convergence behavior of our algorithm (4). In addition, we introduce the following known results and use them to compare the results in Theorem 1.

Theorem 2 (Boikanyo[3]). *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $K : \mathcal{H}_1 \to \mathcal{H}_1$ and $S : \mathcal{H}_2 \to \mathcal{H}_2$ be two demicontractive mappings with coefficients $k_1 \in (-\infty, 1)$ and $k_2 \in (-\infty, 1)$, respectively. Let $I-K$ and $I-S$ be demiclosed at 0. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear operator with the adjoint operator A^*. The iterative sequence $\{z_n\}$ of the split common fixed point problem (1) is generated by the following iterative scheme.*

$$\begin{cases} u_n = z_n - \beta_n A^*(I-S)Az_n, \\ z_{n+1} = \delta_n u + (1-\delta_n)((1-\omega)u_n + \omega K u_n), \ \forall n \geq 1, \end{cases} \tag{9}$$

where $\beta_n = \frac{(1-k_2)\|(I-S)Az_n\|^2}{2\|A^*(I-S)Az_n\|^2}$ with $Az_n \neq SAz_n$; otherwise, $\beta_n = 0$. Meanwhile, $\omega \in (0, 1-k_1)$ and $\delta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\delta_n \to 0$ as $n \to \infty$. If the solution set Ω is nonempty, the iterative sequence $\{z_n\}$ converges strongly to a point $\hat{z} \in \Omega$ that is nearest to u.

Example 1. Let $\mathcal{H}_1 = \mathcal{H}_2 = \mathbb{R}^3$.

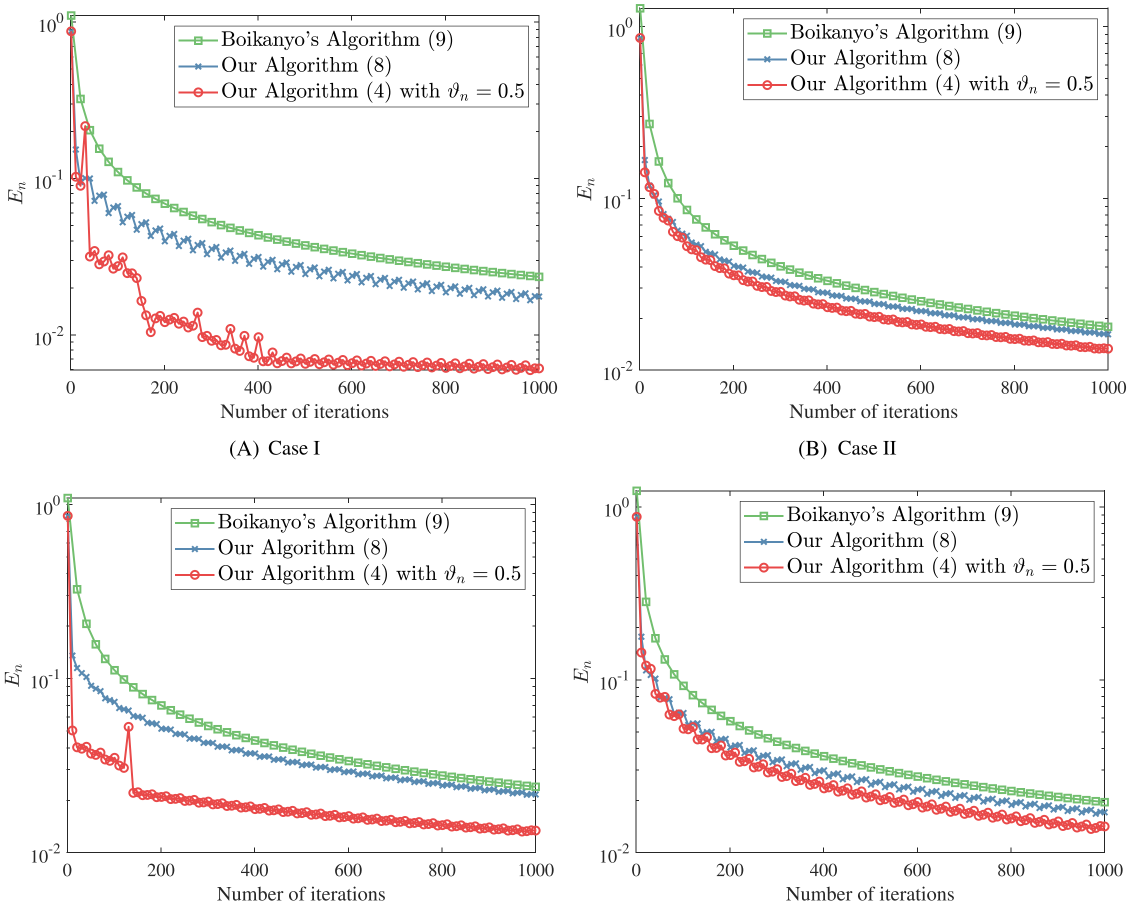
$$C = \{(z_1, z_2, z_3) \in \mathcal{H}_1, z_2^2 + z_3^2 - 1 \leq 0\}$$

and

$$Q = \{(y_1, y_2, y_3) \in \mathcal{H}_2, y_1^2 - y_2 + 5 \leq 0\}.$$

Let $K = P_C : \mathcal{H}_1 \to C$ and $S = P_Q : \mathcal{H}_2 \to Q$ be two metric projection mappings. Let the bounded nonlinear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ be defined by $\begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then, $z^* = (0, 1, 0)$ is a unique solution of split common fixed point problem (1).

Next, we give the relevant parameters in the iterative algorithms. In our algorithm (4), we set $\vartheta_n = 0.5$ and $\sigma_n = 0.5$. In algorithm (9), we take $\delta_n = \frac{1}{n+1}$, $u = z_0$ and $\omega = 0.5$. The error of the iterative algorithms is denoted by $E_n = \|z_n - z^*\|^2$. Take different initial points z_0 and z_1 are generated randomly in MATLAB and maximum iteration 1000 as the stopping criterion. Our numerical results are shown in Figure 1.



(A) Case I

(B) Case II

(C) Case III

(D) Case IV

FIGURE 1 Numerical results for Example 1 [Colour figure can be viewed at wileyonlinelibrary.com]

Example 2. Let $\mathcal{H}_1 = \mathcal{H}_2 = L_2([0, 2\pi])$ with the inner product $\langle z, y \rangle := \int_0^{2\pi} z(t)y(t)\mathrm{d}t$ and with the norm which defined by $\|z\|_2 := \left(\int_0^{2\pi} |z(t)|^2 \mathrm{d}t\right)^{\frac{1}{2}}$, $\forall z, y \in L_2([0, 2\pi])$. Further, we consider the following half-space:

$$C = \left\{ z \in L_2([0, 2\pi]) | \int_0^{2\pi} z(t)\mathrm{d}t \leq 1 \right\} \text{and}\, Q = \left\{ y \in L_2([0, 2\pi]) | \int_0^{2\pi} |y(t) - \sin(t)|^2 \mathrm{d}t \leq 16 \right\}.$$

In addition, a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Az)(t) := z(t)$. Then, $(A^*z)(t) = z(t)$ and $\|A\| = 1$. We can also write the projections onto C and the projections onto Q as follows.

$$P_C(z) = \begin{cases} \frac{1 - \int_0^{2\pi} z(t)\mathrm{d}t}{4\pi^2} + z, & \int_0^{2\pi} z(t)\mathrm{d}t > 1, \\ z, & \int_0^{2\pi} z(t)\mathrm{d}t \leq 1. \end{cases}$$

$$P_Q(y) = \begin{cases} \sin(t) + \frac{4}{\sqrt{\int_0^{2\pi} |y(t) - \sin(t)|^2 \mathrm{d}t}}(y - \sin(t)), & \int_0^{2\pi} |y(t) - \sin(t)|^2 \mathrm{d}t > 16, \\ y, & \int_0^{2\pi} |y(t) - \sin(t)|^2 \mathrm{d}t \leq 16. \end{cases}$$

Now, we solve problem (1), where $K = P_C$ and $S = P_Q$. Choose different initial values z_0 and z_1. The error of the iterative algorithms is denoted by

$$E_n = \frac{1}{2} \|P_C(z_n) - z_n\|_2^2 + \frac{1}{2} \|P_Q(A(z_n)) - A(z_n)\|_2^2.$$

TABLE 1 Numerical results for Example 2

Cases	Initial values	Our algorithm (4) with $\vartheta_n = 0.5$ Iter.	Time (s)	Our algorithm (8) Iter.	Time (s)	Boikanyo's algorithm (9) Iter.	Time (s)
I	$z_0 = \sin(t), z_1 = \frac{t^2}{5}$	57	93.1586	76	119.7086	200	50.4863
II	$z_0 = t^2, z_1 = \frac{e^t}{20}$	40	110.9755	48	77.8733	200	61.2474
III	$z_0 = t^2, z_1 = \frac{2^t}{2}$	73	241.0647	109	225.5353	200	55.4393
IV	$z_0 = 2t^2, z_1 = \frac{t^3}{10}$	73	110.7121	80	145.8766	200	50.3234

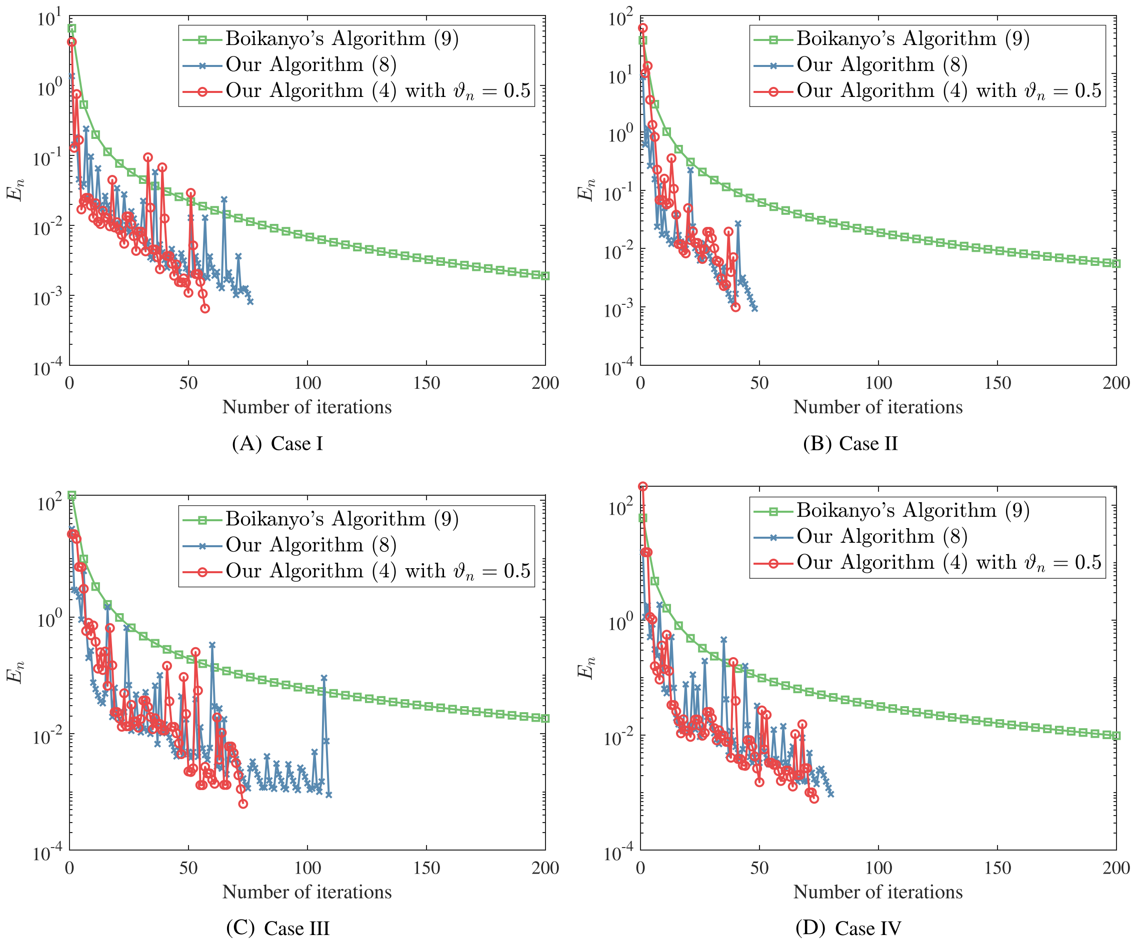


FIGURE 2 Convergence behavior of iteration error $\{E_n\}$ with different initial values for Example 2 [Colour figure can be viewed at wileyonlinelibrary.com]

All parameters are the same as in Example 1. We take the error $E_n < 10^{-3}$ or maximum iteration 200 as the stopping criterion. All numerical results are shown in Table 1 and Figure 2. In Table 1, Iter. and Time (s) denote the number of iterations and the CPU time in seconds, respectively.

Remark 1.

(i) As show in Examples 1 and 2, we see that our algorithm (4) with the inertial term outperforms algorithm (8) and Boikanyo's algorithm (9) in the number of iterations. However, our algorithm (4) has no advantage in CPU time, because in each iteration, we need to calculate the projections onto C_n and Q_n.

(ii) Our proposed algorithm is consistent in the sense that the choice of initial points does not affect the required number of iterations needed to achieve desired results.

5 | CONCLUSION

The important conclusion is that we give an algorithm (i.e., algorithm 4) to approximate the solution of the split common fixed point problem for demicontractive mappings by the inertial technique and the hybrid projection method in Section 3. For better convergence results, we introduce a new self-adaptive step-size sequence which does not need prior knowledge of operator norms. Furthermore, the corresponding strong convergence theorem (i.e., Theorem 1) is proved by such a self-adaptive step-size sequence. In addition, as we can see in Figures 1 and 2. Our results are effective in practice and improve the known results.

ACKNOWLEDGEMENTS

The authors thank the reviewers for their helpful suggestions on our original manuscript.

CONFLICT OF INTEREST

This work has no conflict of interest.

ORCID

Zheng Zhou https://orcid.org/0000-0003-3847-436X
Bing Tan https://orcid.org/0000-0003-1509-1809
Songxiao Li https://orcid.org/0000-0001-7911-2758

REFERENCES

1. Cho SY, Qin X. On the strong convergence of an iterative process for asymptotically strict pseudocontractions and equilibrium problems. *Appl Math Comput*. 2014;235(1):430-438.
2. Wang Y, Liu W, Song Y, Fang X. Mixed iterative algorithms for the multiple-set split equality common fixed-point problem of demicontractive mappings. *J Nonlinear Convex Anal*. 2018;19(11):1921-1932.
3. Boikanyo OA. A strongly convergent algorithm for the split common fixed point problem. *Appl Math Comput*. 2015;265:844-853.
4. Padcharoen A, Kumam P, Cho YJ. Split common fixed point problems for demicontractive operators. *Numer Algorithms*. 2019;82(1):297-320.
5. Suparatulatorn R, Khemphet A, Charoensawan P, Suantai S, Phudolsitthiphat N. Generalized self-adaptive algorithm for solving split common fixed point problem and its application to image restoration problem. *Int J Comput Math*. 2020;97(7):1431-1443.
6. Yao Y, Liou YC, Postolache M. Self-adaptive algorithms for the split problem of the demicontractive operators. *Optimization*. 2018;67(9):1309-1319.
7. Censor Y, Elfving T. A multiprojection algorithm using Bregman projections in a product space. *Numer Algorithms*. 1994;8(2):221-239.
8. Censor Y, Segal A. The split common fixed point problem for directed operators. *J Convex Anal*. 2009;16(2):587-600.
9. Moudafi A. A note on the split common fixed-point problem for quasi-nonexpansive operators. *Nonlinear Anal*. 2011;74(12):4083-4087.
10. Moudafi A. The split common fixed-point problem for demicontractive mappings. *Inverse Probl*. 2010;26(5):55007.
11. Takahashi W, Yao JC. The split common fixed point problem for two finite families of nonlinear mappings in Hilbert spaces. *J Nonlinear Convex Anal*. 2019;20(2):173-195.
12. Qin X, Wang L, Yao JC. Inertial splitting method for maximal monotone mappings. *J Nonlinear Convex Anal*. 2020;21(10):2325-2333.
13. Cui H, Wang F. Iterative methods for the split common fixed point problem in Hilbert space. *Fixed Point Theory Appl*. 2014;2014(1):78.
14. Vinh NT, Hoai PT. Some subgradient extragradient type algorithms for solving split feasibility and fixed point problems. *Math Meth Appl Sci*. 2016;39(13):3808-3823.
15. Liu L. A hybrid steepest descent method for solving split feasibility problems involving nonexpansive mappings. *J Nonlinear Convex Anal*. 2019;20(3):471-488.
16. Zhou Z, Tan B, Li S. An inertial shrinking projection algorithm for split common fixed point problems. *J Appl Anal Comput*. 2020a;10(5):2104-2120.
17. Zhou Z, Tan B, Li S. A new accelerated self-adaptive stepsize algorithm with excellent stability for split common fixed point problems. *Comput Appl Math*. 2020b;39(3):220.
18. Masad E, Reich S. A note on the multiple-set split convex feasibility problem in Hilbert space. *J Nonlinear Convex Anal*. 2007;8(3):367-371.
19. Reich S, Tuyen TM. Two projection algorithms for solving the split common fixed point problem. *J Optim Theory Appl*. 2020;186(1):148-168.
20. Reich S, Tuyen TM, Trang NM. Parallel iterative methods for solving the split common fixed point problem in Hilbert spaces. *Numer Funct Anal Optim*. 2020;41(7):778-805.
21. Kraikaew R, Saejung S. On split common fixed point problems. *J Math Anal Appl*. 2014;415(2):513-524.
22. Qin X, Yao JC. A viscosity iterative method for a split feasibility problem. *J Nonlinear Convex Anal*. 2019;20(8):1497-1506.

23. Wang F. A new iterative method for the split common fixed point problem in Hilbert spaces. *Optimization*. 2017;66(3):407-415.
24. Nakajo K, Takahashi W. Strong convergence theorems for nonexpansive mappings and nonexpansive semigroups. *J Math Anal Appl*. 2003;279(2):372-379.
25. Tan B, Xu S, Li S. Inertial shrinking projection algorithms for solving hierarchical variational inequality problems. *J Nonlinear Convex Anal*. 2020;21(4):871-884.
26. Alvarez F. Attouch H. An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping. *Set-Valued Var Anal*. 2001;9(1–2):3-11.
27. Tan B, Zhou Z, Qin X. Accelerated projection-based forward-backward splitting algorithms for monotone inclusion problems. *J Appl Anal Comput*. 2020;10(5):2184-2197.
28. Boţ TI. Inertial Douglas-Rachford splitting for monotone inclusion problems. *Appl Math Comput*. 2015;256:472-487.
29. Tan B, Xu S. Strong convergence of two inertial projection algorithms in Hilbert spaces. *J Appl Numer Optim*. 2020;2(2):171-186.
30. Tan B, Fan J, Li S. Self-adaptive inertial extragradient algorithms for solving variational inequality problems. *Comput Appl Math*. 2021;40(1):19.
31. Fan J, Liu L, Qin X. A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities. *Optimization*. 2020;69(9):2199-2215.
32. Shehu Y, Iyiola OS, Ogbuisi FU. Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing. *Numer Algorithms*. 2020;83(4):1321-1347.
33. Tan B, Zhou Z, Li S. Strong convergence of modified inertial Mann algorithms for nonexpansive mappings. *Mathematics*. 2020;8(4):462.
34. Goebel K, Reich S. *Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings*: Marcel Dekker; 1984.
35. Kopecká E, Reich S. A note on alternating projections in Hilbert space. *J Fixed Point Theory Appl*. 2012;12(1–2):41-47.
36. Takahashi W. *Introduction to Nonlinear and Convex Analysis*: Yokohama Publishers; 2009.
37. Zhou H, Qin X. *Fixed Points of Nonlinear Operators: Iterative Methods*: De Gruyter; 2020.
38. Marino G, Xu HK. Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces. *J Math Anal Appl*. 2007;329(1):336-346.

How to cite this article: Zhou Z, Tan B, Li S. An accelerated hybrid projection method with a self-adaptive step-size sequence for solving split common fixed point problems. *Math Meth Appl Sci*. 2021;44:7294–7303. https://doi.org/10.1002/mma.7261