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A new accelerated self-adaptive stepsize algorithm with excellent stability for split common fixed point problems

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Abstract
In the framework of Hilbert spaces, we study the solutions of split common fixed point problems. A new accelerated self-adaptive stepsize algorithm with excellent stability is proposed under the effects of inertial techniques and Meir–Keeler contraction mappings. The strong convergence theorems are obtained without prior knowledge of operator norms. Finally, in applications, our main results in this paper are applied to signal recovery problems.

Keywords Self-adaptive stepsize · Meir–Keeler contraction · Inertial technique · Signal recovery

Mathematics Subject Classification 47H10 · 47J25 · 65K10 · 65Y10

1 Introduction

Based on the idea of the split feasibility problem (for short, SFP), Censor and Segal (2009) introduced the split common fixed point problem (for short, SCFPP) in 2009 as follows. Let H_1 and H_2 be Hilbert spaces, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear mappings, $F(T)$ and $F(S)$ denote the fixed point sets of T and S, respectively, and $A : H_1 \to H_2$ be a bounded linear operator. The split common fixed point problem is to find:

$$x^* \in F(T) \quad \text{and} \quad Ax^* \in F(S). \tag{1.1}$$

Under certain conditions, when $T = P_C$ and $S = P_Q$ (P_C and P_Q are metric projections from H_1 to its nonempty closed convex subset C and H_2 to its nonempty convex closed subset Q,

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respectively). The problem (1.1) can be considered as the split feasibility problems, which is to find a point $x^* \in C$ and $Ax^* \in Q$. This problem was introduced by Censor and Elfving (1994).

Naturally, x^* is a solution of the split common fixed point problem if and only if x^* is a solution of the equation $x^* = T(I - \beta A^*(I - S)A)x^*$, where A^* is an adjoint operator of A. Furthermore, Censor and Segal (2009) proposed an algorithm to approximate a solution of the problem (1.1) in finite-dimensional Euclidean spaces by the recursive procedure: $x_{n+1} = T(I - \beta A^{\mathrm{T}}(I - S)A)x_n$, where T and S are directed operators, A^{T} is the matrix transposition of A, M is the largest eigenvalue of matrix $A^{\mathrm{T}}A$ and $\beta \in (0, 2/M)$. By the proposed algorithm, they obtained that the iterative sequence $\{x_n\}$ converges to a solution of the problem (1.1). In addition, the split feasibility problem and the split common fixed point problem have been widely studied in many mathematical problems, such as variational inequality problems, equilibrium problems, monotone inclusion problems, etc. (see Censor et al. 2012; Cho and Kang 2012; Chang et al. 2018; Majee and Nahak 2018; Shehu and Agbebaku 2018; Qin and Yao 2019). They are also applied to many real problems, such as medical imaging, astronomy, compressed sensing, radiation therapy treatment planning, and remote sensing (see Chambolle and Lions 1997; Nikolova 2004; Qin and An 2019; An et al. 2020). Furthermore, such an algorithm in Censor and Segal (2009) was widely extended to various operators, such as quasi-nonexpansive operators and demicontractive operators (see Moudafi 2010, 2011; Cui and Wang 2014). Unfortunately, the weak convergence results of these studies were only guaranteed.

To obtain strong convergence properties, some algorithms were considered by combining Halpern algorithms and viscosity algorithms under mild conditions, see, for example, Boikanyo (2015), Kraikaew and Saejung (2014), and He et al. (2016). The viscosity algorithm was introduced by Moudafi (2000) and the strong convergence was obtained using contraction mappings. Before that, Meir and Keeler (1969) introduced Meir–Keeler contraction as follows. Let (X, d) be a metric space. $\xi : X \to X$ is a Meir–Keeler contraction mapping if and only if, for each $\varepsilon > 0$, there exists a number $\delta > 0$, such that $d(x, y) < \varepsilon + \delta \Rightarrow d(\xi(x), \xi(y)) < \varepsilon, \forall x, y \in X$. Obviously, the contraction mapping can also be regarded as a special case of the Meir–Keeler contraction mapping. On the contrary, it is not true. After this, the Meir–Keeler contraction mapping is also widely studied (for more details, see Suzuki 2007; Karpagam and Agrawal 2011; Vaish and Ahmad 2020 and the references therein). As a promotion, Wang (2017) and Yao et al. (2018) studied the new iterative algorithm $x_{n+1} = x_n - \beta_n[(I - T) + A^*(I - S)A)]x_n,\ \forall n \geq 1$, where $\{\beta_n\}$ is a self-adaptive stepsize sequence. On the other hand, to achieve better convergence rate of iterative algorithms, the inertial effects have been studied recently in Alvarez and Attouch (2001), Maingé and Moudafi (2008) and the references therein. Based on the work in Alvarez and Attouch (2001), Meir and Keeler (1969), Wang (2017), and Yao et al. (2018), we will consider the following questions.

Can we combine the inertial technique and the Meir–Keeler contraction to build a new iterative algorithm for solving the problem (1.1)? *Can we find a self-adaptive stepsize sequence to ensure the effectiveness of this algorithm?*

For these questions, in this paper, we come up with a new modified algorithm by the inertial technique and the Meir–Keeler contraction in the framework of infinite Hilbert spaces. The strong convergence of this algorithm for the problem (1.1) is obtained without prior knowledge of operator norms. It is worth noting that excellent stability and better convergence rate of our algorithm are guaranteed by the proposed self-adaptive stepsize. Furthermore, some numerical experiments are used to demonstrate and show the efficiency of our main results.

The organization of this paper is as follows. Some basic properties and relevant lemmas are introduced in Sect. 2 and used in the proof for the main results of this paper in Sect. 3. Some theoretical applications are also proposed in Sect. 4. Finally, in Sect. 5, some numerical experiments demonstrate the validity and authenticity of our results.

2 Preliminaries

For the convenience and standard in the rest of this paper, the notations $\to$ and $\rightharpoonup$ denote strong convergence and weak convergence, respectively. The fixed point set of a mapping T is marked as $F(T)$. Some well-known basic properties are stated as follows.

(P1) The metric projection from H onto C is represented by P_C, that is:

$$P_C x = \operatorname{argmin}_{y \in C} \|x - y\|, \quad \forall x \in H.$$

It also have the following equivalent forms:

$$\langle P_C x - x, P_C x - y \rangle \le 0, \forall y \in C \Leftrightarrow \|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x - y\|^2.$$

(P2) The mapping $T : H \to H$ and $F(T) \neq \emptyset$. $I - T$ is demiclosed at zero if and only if $\forall \{x_n\} \subset H$, $\{x_n\}$ converges weakly to x, and $(I - T)x_n$ converges strongly to 0, then $x \in F(T)$.

(P3) For any $x, y \in H$, the following properties hold:

$$\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y \rangle \le \|x\|^2 + 2\langle y, x + y \rangle,$$
$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \quad \forall \lambda \in \mathbb{R}.$$

A mapping $T : H \to H$ is said to be:

- contraction if there exists a constant $\alpha \in [0, 1)$, such that:

$$\|Tx - Ty\| \le \alpha\|x - y\|, \ \forall x, y \in H.$$

- nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H.$$

- quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\|, \ \forall x \in H, \ p \in F(T).$$

- strictly pseudo-contractive if there exists $k \in [0, 1)$, such that:

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + k\|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in H.$$

- pseudo-contractive if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|x - Tx - (y - Ty)\|^2, \quad \forall x, y \in H.$$

- directed if $F(T) \neq \emptyset$ and

$$\|Tx - p\|^2 \le \|x - p\|^2 - \|Tx - x\|^2, \quad \forall x \in H, \ p \in F(T).$$

- demicontractive if there exists a number $k \in (-\infty, 1)$ and $F(T) \neq \emptyset$, such that:

$$\|Tx - p\|^2 \le \|x - p\|^2 + k\|Tx - x\|^2,$$

or equivalently

$$\langle x - p, x - Tx \rangle \geq \frac{1-k}{2} \|x - Tx\|^2.$$

Remark 2.1 From the definition of the mentioned mappings, we get the following relationships:

$$\begin{aligned} &\text{contraction} \Rightarrow \text{nonexpansive} \Rightarrow \text{strictly pseudo-contractive} \\ &\quad \Rightarrow \text{pseudo-contractive.} \end{aligned}$$

If $F(T) \neq \emptyset$, the following relationships hold:

$$\begin{array}{ccc} \text{pseudo-contractive} & \longrightarrow & \text{demicontractive} \\ \uparrow & & \uparrow \\ \text{strictly pseudo-contractive} & \longleftarrow & \text{directed.} \end{array}$$

Here, we give an example of Meir–keeler contraction mapping as follows.

Example 2.1 Let $X = [0, 1] \bigcup \{2, 3, 4, 5, \ldots, 2n, 2n+1, \ldots\}$ with Euclidean distance. Define a mapping $f : X \to \mathbb{R}$ by:

$$f(x) = \begin{cases} \frac{x}{3}, & 0 \leq x \leq 1, \\ 0, & x = 2n (n = 1, 2, \ldots), \\ 1 - \frac{1}{n}, & x = 2n + 1 (n = 1, 2, \ldots). \end{cases}$$

By the definition of f above, it can be seen that f is not a contraction (since a number $\alpha \in [0, 1)$ in the definition of the contraction mapping cannot be found). Besides, for any $\varepsilon \geq 1$ and $\delta > 0$, i.e., $|x - y| < \varepsilon + \delta$, we easily know that $|f(x) - f(y)| < \varepsilon, \forall x, y \in X$. On the other hand, for $0 < \varepsilon < 1$, put $\delta \in (0, \min\{1 - \varepsilon, 2\varepsilon\})$, i.e., $|x - y| < \varepsilon + \delta$, we have $|f(x) - f(y)| < \varepsilon$. In summary, the mapping f is a Meir–keeler contraction.

Lemma 2.1 (Meir and Keeler 1969) *Let B be a Banach space and C be a closed convex subset of B. $\xi : C \to C$ is a Meir–Keeler contraction mapping if and only if for any $\varepsilon > 0$, there exists a number $\alpha_\varepsilon > 0$, such that $\|x - y\| \geq \varepsilon$ implies $\|\xi(x) - \xi(y)\| \leq \alpha_\varepsilon \|x - y\|$.*

Lemma 2.2 (Takahashi 2017) *Let H be a Hilbert space and C be a closed convex subset of H, $T : C \to H$ be a demicontractive mapping with $k \in (-\infty, 1)$. Then, $F(T)$ is closed and convex.*

Lemma 2.3 (Marino and Xu 2007; Zhou 2008) *Let C be a nonempty closed convex subset of a Hilbert space H, and $T : C \to H$ be a strictly pseudo-contractive mapping with coefficient $k \in [0, 1)$. $F(T)$ is closed and convex, and $I - T$ is demiclosed at* 0.

Lemma 2.4 (He and Yang 2013) *Let $\{\Delta_n\}$ and $\{\mu_n\}$ be two non-negative real numbers sequences, such that:*

$$\begin{aligned} \Delta_{n+1} &\leq (1 - \delta_n)\Delta_n + \delta_n \vartheta_n, \quad n \geq 1, \\ \Delta_{n+1} &\leq \Delta_n - \mu_n + \zeta_n, \quad n \geq 1, \end{aligned}$$

where $\{\vartheta_n\}$, $\{\zeta_n\}$, and $\{\delta_n\}$ are real sequences with $0 < \delta_n < 1$. If $\sum_{n=1}^{\infty} \delta_n = \infty$, $\lim_{n\to\infty} \zeta_n = 0$, and $\lim_{k\to\infty} \mu_{n_k} = 0$ implies $\limsup_{k\to\infty} \vartheta_{n_k} \leq 0$ ($\{n_k\}$ is any real number subsequence of $\{n\}$). The sequence $\{\Delta_n\}$ converges to 0 *as $n \to \infty$.*

3 Self-adaptive inertial Meir–Keeler contraction algorithms

In this section, we propose an algorithm to approximate a solution of the problem (1.1), and assume that its solution set is nonempty, i.e., $\Omega = \{x^* : x^* \in F(T), Ax^* \in F(S)\} \neq \emptyset$. In addition, the following assumptions are presupposed.

(A1) H_1 and H_2 are two Hilbert spaces;
(A2) $A : H_1 \to H_2$ is a bounded linear operator with adjoint operator A^*;
(A3) $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are demicontractive mappings with coefficients $k_1 \in (-\infty, 1)$ and $k_2 \in (-\infty, 1)$, respectively;
(A4) $\xi : H_1 \to H_1$ is a Meir–Keeler contraction mapping.

The iterative sequence $\{x_n\}$ of the split common fixed point problem is generated by the following recursive procedure.

Algorithm 1: Self-adaptive inertial Meir–Keeler contraction algorithm (SIMKCA)

Input: Initial points $x_0, x_1 \in H_1$.
Step 1 For any $n \geq 1$, the inertial parameters $\tau_n \in [0, 1)$ with $\lim_{n\to\infty} \frac{\tau_n}{\theta_n} \|x_n - x_{n-1}\| = 0$. Compute

$$w_n = x_n + \tau_n(x_n - x_{n-1}); \tag{3.1}$$

Step 2 If $(I - S)Aw_n \neq 0$, the stepsize $\beta_n = \sigma_n \min\left\{\frac{1-k_1}{2}, \frac{(1-k_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$. Otherwise, $\beta_n = \sigma_n(1 - k_1)/2$. Compute:

$$u_n = w_n - \beta_n\left[(I - T)w_n + A^*(I - S)Aw_n\right]; \tag{3.2}$$

Step 3 Put $\theta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$. Compute:

$$x_{n+1} = \theta_n \xi(u_n) + (1 - \theta_n)u_n. \tag{3.3}$$

Remark 3.1 1. In numerical computation, the value of $\|x_n - x_{n-1}\|$ is known in each iteration, and the inertial parameter sequence $\{\tau_n\}$ is easily implemented in Step 1 of Algorithm 1. For example, τ_n can be selected by the following form:

$$\tau_n = \begin{cases} \min\left\{\tau, \dfrac{\alpha_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \tau, & \text{otherwise}, \end{cases}$$

where $\{\alpha_n\}$ is a positive sequence with $\alpha_n = o(\theta_n)$ and $\tau \in [0, 1)$.
2. In addition, since the parameter limitation in Step 3 of Algorithm 1, the sequence $\{\theta_n\}$ can be chosen by $\theta_n = \frac{1}{n^p}$ $(0 < p \leq 1)$. Furthermore, using the choice of τ_n above, we can easily choose $\alpha_n = \frac{1}{n^q}$ $(q > p)$.

Theorem 3.1 *Assumed that (A1)–(A4) hold, $I - T$ and $I - S$ are demiclosed at* 0*, the iterative sequence $\{x_n\}$ generated by* Algorithm 1 *converges strongly to $\hat{x} = P_\Omega \xi(\hat{x}) \in \Omega$.*

Proof **Step 1** First, it follows from Lemma 2.2 that $F(T)$ and $F(S)$ are nonempty closed convex. This implies that Ω is closed convex and P_Ω is well defined. Take any $\hat{x} = P_\Omega \xi(\hat{x}) \in \Omega$, that is, $\hat{x} \in F(T)$ and $A\hat{x} \in F(S)$. For any $\varepsilon > 0$, $\{x_n\}$ is bounded via $\|x_n - \hat{x}\| \leq \varepsilon$. On the contrary, for $\|x_n - \hat{x}\| \geq \varepsilon$, it follows from Lemma 2.1 that there exists a number $\alpha_\varepsilon \in (0, 1)$ about ε, such that:

$$\|\xi(x_n) - \xi(\hat{x})\| \leq \alpha_\varepsilon \|x_n - \hat{x}\|.$$

From (3.2), we get:

$$\begin{aligned}\|u_n-\hat{x}\|^2 &= \|w_n-\hat{x}\|^2 - 2\beta_n\langle (I-T)w_n + A^*(I-S)Aw_n, w_n-\hat{x}\rangle \\ &\quad + \beta_n^2\|(I-T)w_n + A^*(I-S)Aw_n\|^2 \\ &= \|w_n-\hat{x}\|^2 - 2\beta_n\langle (I-T)w_n, w_n-\hat{x}\rangle - 2\beta_n\langle (I-S)Aw_n, Aw_n - A\hat{x}\rangle \\ &\quad + \beta_n^2\|(I-T)w_n + A^*(I-S)Aw_n\|^2 \\ &\le \|w_n-\hat{x}\|^2 - \beta_n(1-k_1)\|(I-T)w_n\|^2 - \beta_n(1-k_2)\|(I-S)Aw_n\|^2 \\ &\quad + 2\beta_n^2\left(\|(I-T)w_n\|^2 + \|A^*(I-S)Aw_n\|^2\right). \end{aligned} \tag{3.4}$$

From the definition of $\{\beta_n\}$, we have $(1-k_1-2\beta_n)\|(I-T)w_n\|^2 \ge 0$ and $(1-k_2)\|(I-S)Aw_n\|^2 - 2\beta_n\|A^*(I-S)Aw_n\|^2 \ge 0$. Hence, we set:

$$\Lambda_n = (1-k_1-2\beta_n)\|(I-T)w_n\|^2 + (1-k_2)\|(I-S)Aw_n\|^2 - 2\beta_n\|A^*(I-S)Aw_n\|^2.$$

Furthermore, we have $\Lambda_n \ge 0$ and:

$$\|u_n-\hat{x}\|^2 \le \|w_n-\hat{x}\|^2 - \beta_n\Lambda_n \le \|w_n-\hat{x}\|^2.$$

By (3.1), (3.3), and (3.4), we get:

$$\begin{aligned}\|x_{n+1}-\hat{x}\| &\le \theta_n\|f(u_n)-\hat{x}\| + (1-\theta_n)\|u_n-\hat{x}\| \\ &\le \theta_n\|f(u_n)-f(\hat{x})\| + \theta_n\|f(\hat{x})-\hat{x}\| + (1-\theta_n)\|u_n-\hat{x}\| \\ &\le (1-\theta_n(1-\alpha_\varepsilon))\|u_n-\hat{x}\| + \theta_n\|f(\hat{x})-\hat{x}\| \\ &\le (1-\theta_n(1-\alpha_\varepsilon))\|w_n-\hat{x}\| + \theta_n\|f(\hat{x})-\hat{x}\| \\ &\le (1-\theta_n(1-\alpha_\varepsilon))\|x_n-\hat{x}\| + \theta_n\|f(\hat{x}) \\ &\quad -\hat{x}\| + (1-\theta_n(1-\alpha_\varepsilon))\tau_n\|x_n-x_{n-1}\| \\ &\le (1-\theta_n(1-\alpha_\varepsilon))\|x_n-\hat{x}\| + \theta_n(1-\alpha_\varepsilon) \\ &\quad \times\left(\frac{\|f(\hat{x})-\hat{x}\|}{1-\alpha_\varepsilon} + \frac{\tau_n\|x_n-x_{n-1}\|}{\theta_n(1-\alpha_\varepsilon)}\right). \end{aligned} \tag{3.5}$$

From the conditions $\lim_{n\to\infty}\frac{\tau_n}{\theta_n}\|x_n-x_{n-1}\| = 0$ and $0<\alpha_\varepsilon<1$, we have:

$$\lim_{n\to\infty}\frac{\tau_n\|x_n-x_{n-1}\|}{\theta_n(1-\alpha_\varepsilon)} = \lim_{n\to\infty}\frac{1}{1-\alpha_\varepsilon}\cdot\frac{\tau_n}{\theta_n}\|x_n-x_{n-1}\| = 0.$$

Therefore, there exists a non-negative constant $G>0$, such that $G/2 = \max\left\{\frac{\|f(\hat{x})-\hat{x}\|}{1-\alpha_\varepsilon}, \frac{\tau_n\|x_n-x_{n-1}\|}{\theta_n(1-\alpha_\varepsilon)}\right\}$. By virtue of (3.5), we obtain:

$$\begin{aligned}\|x_{n+1}-\hat{x}\| &\le (1-\theta_n(1-\alpha_\varepsilon))\|x_n-\hat{x}\| + \theta_n(1-\alpha_\varepsilon)G \\ &\le \max\{\|x_n-\hat{x}\|, G\} \le \cdots \le \max\{\|x_0-\hat{x}\|, G\}.\end{aligned}$$

This implies that $\{x_n\}$ is bounded. Similarly, we also have that $\{u_n\}$ and $\{w_n\}$ are bounded.

Step 2 According to (3.1) and the property (P3), we have:

$$\begin{aligned}\|w_n-\hat{x}\|^2 = \|x_n+\tau_n(x_n-x_{n-1})-\hat{x}\|^2 &\le \|x_n-\hat{x}\|^2+2\tau_n\langle w_n-\hat{x}, x_n-x_{n-1}\rangle \\ &\le \|x_n-\hat{x}\|^2+2\tau_n\|w_n-\hat{x}\|\|x_n-x_{n-1}\|.\end{aligned} \tag{3.6}$$

On the other hand, using the property (P3), (3.3), (3.4), and (3.6), we can obtain:

$$\begin{aligned}\|x_{n+1}-\hat{x}\|^2 &\le \|\theta_n(f(u_n)-f(\hat{x}))+(1-\theta_n)(u_n-\hat{x})\|^2+2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle \\ &\le \theta_n\|f(u_n)-f(\hat{x})\|^2+(1-\theta_n)\|u_n-\hat{x}\|^2+2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle \\ &\le (1-\theta_n(1-\alpha_\varepsilon^2))\|u_n-\hat{x}\|^2+2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle \\ &\le (1-\theta_n(1-\alpha_\varepsilon^2))\|w_n-\hat{x}\|^2-(1-\theta_n(1-\alpha_\varepsilon^2))\beta_n\Lambda_n \\ &\quad +2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle \\ &\le (1-\theta_n(1-\alpha_\varepsilon^2))\|x_n-\hat{x}\|^2+2(1-\theta_n(1-\alpha_\varepsilon^2))\tau_n\|w_n-\hat{x}\|\|x_n-x_{n-1}\| \\ &\quad +2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle-(1-\theta_n(1-\alpha_\varepsilon^2))\beta_n\Lambda_n.\end{aligned}$$

For each $n\ge 1$, we set:

$$\begin{aligned}&\Delta_n=\|x_n-\hat{x}\|^2,\ \delta_n=\theta_n(1-\alpha_\varepsilon^2),\ \mu_n=(1-\theta_n(1-\alpha_\varepsilon^2))\beta_n\Lambda_n; \\ &\vartheta_n=\frac{2(1-\delta_n)\tau_n\|w_n-\hat{x}\|\|x_n-x_{n-1}\|+2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle}{\theta_n(1-\alpha_\varepsilon^2)}; \\ &\zeta_n=2(1-\delta_n)\tau_n\|w_n-\hat{x}\|\|x_n-x_{n-1}\|+2\theta_n\langle f(\hat{x})-\hat{x}, x_{n+1}-\hat{x}\rangle.\end{aligned}$$

Then, the above formula is reduced to the following inequalities:

$$\Delta_{n+1}\le(1-\delta_n)\Delta_n+\delta_n\vartheta_n \ and\ \Delta_{n+1}\le\Delta_n-\mu_n+\zeta_n, n\ge 1.$$

By the boundedness of $\{x_n\}$ and $\{w_n\}$, $\sum_{n=1}^{\infty}\theta_n=\infty$, $\lim_{n\to\infty}\theta_n\to 0$, $\lim_{n\to\infty}\frac{\tau_n}{\theta_n}\|x_n-x_{n-1}\|=0$ and $0<\alpha_\varepsilon<1$, we obtain $\lim_{n\to\infty}\zeta_n=0$ and $\sum_{n=1}^{\infty}\delta_n=\infty$. By Lemma 2.4, we need to show that $\lim_{k\to\infty}\mu_{n_k}=0$ implies $\limsup_{k\to\infty}\vartheta_{n_k}\le 0$ for any subsequence of real numbers $\{n_k\}$ of $\{n\}$. Let $\{\mu_{n_k}\}$ be a any subsequence of $\{\mu_n\}$, such that $\lim_{k\to\infty}\mu_{n_k}=0$. If $(I-S)Aw_{n_k}\ne 0$, it follows from $\mu_n=(1-\theta_n(1-\alpha_\varepsilon^2))\beta_n\Lambda_n$ that:

$$\lim_{k\to\infty}\|(I-T)w_{n_k}\|=\lim_{k\to\infty}\|(I-S)Aw_{n_k}\|=0. \tag{3.7}$$

By the boundedness of $\{x_n\}$, there exists a sequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, such that $x_{n_{k_j}}\rightharpoonup\bar{x}$, and:

$$\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x}, x_{n_k}-\hat{x}\rangle=\lim_{j\to\infty}\langle f(\hat{x})-\hat{x}, x_{n_{k_j}}-\hat{x}\rangle.$$

In addition, by virtue of $\|w_n-x_n\|=\tau_n\|x_n-x_{n-1}\|\to 0$, we have $\{w_{n_{k_j}}\}\rightharpoonup\bar{x}$. Since A is a bounded linear operator, $Aw_{n_{k_j}}\rightharpoonup A\bar{x}$. Since $I-T$ and $I-S$ are demiclosed at 0, from (3.7), we have $\bar{x}\in F(T)$ and $A\bar{x}\in F(S)$, which implies that $\bar{x}\in\Omega$. On the other hand, if $(I-S)Aw_{n_k}=0$, it is clearly that we can also get the same result as above. In addition, it follows from the property of projection that $\limsup_{k\to\infty}\langle f(\hat{x})-\hat{x}, x_{n_k}-\hat{x}\rangle=\lim_{j\to\infty}\langle f(\hat{x})-\hat{x}, x_{n_{k_j}}-\hat{x}\rangle=\langle f(\hat{x})-\hat{x}, \bar{x}-\hat{x}\rangle\le 0$. According to conditions for parameters θ_n and τ_n, and (3.7), we have:

$$\begin{aligned}\|u_{n_k}-x_{n_k}\| &\le \|w_{n_k}-x_{n_k}\|+\beta_n(\|(I-T)w_{n_k}\|+\|(I-S)Aw_{n_k}\|) \\ &= \tau_n\|x_{n_k}-x_{n_k-1}\|+\beta_n(\|(I-T)w_{n_k}\|+\|(I-S)Aw_{n_k}\|)\to 0, \\ \|x_{n_k+1}-x_{n_k}\| &= \theta_{n_k}\|f(u_{n_k})-x_{n_k}\|+(1-\theta_{n_k})\|u_{n_k}-x_{n_k}\|\to 0.\end{aligned}$$

Hence, we have $\limsup_{k\to\infty} \langle f(\hat{x}) - \hat{x}, x_{n_k+1} - \hat{x}\rangle \le 0$ and:

$$\lim_{n\to\infty} \frac{2(1-\delta_n)\tau_n \|w_n - \hat{x}\| \|x_n - x_{n-1}\|}{\theta_n(1-\alpha_\varepsilon^2)} \le \lim_{n\to\infty} \frac{2\tau_n \|x_n - x_{n-1}\|}{\theta_n} \cdot \frac{\|w_n - \hat{x}\|}{1-\alpha_\varepsilon^2} = 0,$$

which implies that $\limsup_{k\to\infty} \vartheta_{n_k} \le 0$. By Lemma 2.4, we obtain $\lim_{n\to\infty} \Delta_n = 0$, that is, $x_n \to \hat{x}$. □

From Remark 2.1, we obtain a few special cases of demicontractive mappings and got the following corollaries, which also generalize the existing results in Censor and Segal (2009), Cui and Wang (2014), Moudafi (2010, 2011), Wang (2017) and Yao et al. (2018).

If T and S are the strictly pseudo-contractive mappings with coefficients $k_1 \in [0, 1)$ and $k_2 \in [0, 1)$, respectively, and the fixed point sets $F(T)$ and $F(S)$ are nonempty, in other words, T and S are the demicontractive mappings with coefficients $k_1 \in [0, 1)$ and $k_2 \in [0, 1)$, respectively. According to Lemma 2.3, we know that $I - T$ and $I - S$ are demiclosed at 0 and its fixed point sets are closed and convex. Thus, we obtain the following corollary.

Corollary 3.1 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*, $\xi : H_1 \to H_1$ be a Meir–Keeler contraction mapping, and $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be strictly pseudo-contractive mappings with coefficients $k_1 \in [0, 1)$ and $k_2 \in [0, 1)$, respectively. For any initial points $x_0, x_1 \in H_1$, the iterative sequence $\{x_n\}$ is generated by the following cyclic process:*

$$\begin{cases} w_n = x_n + \tau_n(x_n - x_{n-1}), \\ u_n = w_n - \beta_n\left[(I-T)w_n + A^*(I-S)Aw_n\right], \\ x_{n+1} = \theta_n\xi(u_n) + (1-\theta_n)u_n, \quad \forall n \ge 1, \end{cases} \tag{3.8}$$

where if $(I - S)Aw_n \ne 0$, the self-adaptive stepsize $\beta_n = \sigma_n \min\left\{\frac{1-k_1}{2}, \frac{(1-k_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$; otherwise, $\beta_n = \sigma_n(1 - k_1)/2$. Furthermore, $\tau_n \in [0, 1)$, $\lim_{n\to\infty} \frac{\tau_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\theta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$.

If the fixed point sets $F(T)$ and $F(S)$ are nonempty, the iterative sequence $\{x_n\}$ converges strongly to $\hat{x} = P_\Omega \xi(\hat{x}) \in \Omega$.

If the demicontractive mappings T and S with coefficients $k_1 = -1$, $k_2 = -1$, respectively, that is, T and S are the directed mappings. The following corollary holds.

Corollary 3.2 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*, $\xi : H_1 \to H_1$ be a Meir–Keeler contraction mapping, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be directed mappings, and its fixed point sets be nonempty. For any initial points $x_0, x_1 \in H_1$, the iterative sequence $\{x_n\}$ is generated by the following cyclic process:*

$$\begin{cases} w_n = x_n + \tau_n(x_n - x_{n-1}), \\ u_n = w_n - \beta_n\left[(I-T)w_n + A^*(I-S)Aw_n\right], \\ x_{n+1} = \theta_n\xi(u_n) + (1-\theta_n)u_n, \ \forall n \ge 1, \end{cases} \tag{3.9}$$

where if $(I - S)Aw_n \ne 0$, the self-adaptive stepsize $\beta_n = \sigma_n \min\left\{1, \frac{\|(I-S)Aw_n\|^2}{\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$; otherwise, $\beta_n = \sigma_n$. Furthermore, $\tau_n \in [0, 1)$, $\lim_{n\to\infty} \frac{\tau_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\theta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$.

If $I - T$ and $I - S$ are demiclosed at 0*, the sequence* $\{x_n\}$ *converges strongly to* $\hat{x} = P_\Omega \xi(\hat{x}) \in \Omega$.

If the demicontractive mappings T and S with coefficients $k_1 = 0$ and $k_2 = 0$, respectively; that is, T and S are the quasi-nonexpansive mappings. The following corollary holds.

Corollary 3.3 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*, $\xi : H_1 \to H_1$ be a Meir–Keeler contraction mapping, $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be quasi-nonexpansive mappings, and $F(T)$ and $F(S)$ be nonempty. For any initial points $x_0, x_1 \in H_1$, the sequence $\{x_n\}$ is generated by the following algorithm:*

$$\begin{cases} w_n = x_n + \tau_n(x_n - x_{n-1}), \\ u_n = w_n - \beta_n \left[(I - T)w_n + A^*(I - S)Aw_n\right], \\ x_{n+1} = \theta_n \xi(u_n) + (1 - \theta_n)u_n, \ \forall n \geq 1, \end{cases} \tag{3.10}$$

where if $(I - S)Aw_n \neq 0$, the self-adaptive stepsize $\beta_n = \sigma_n \min\left\{\frac{1}{2}, \frac{\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$; otherwise, $\beta_n = \sigma_n/2$. Furthermore, $\tau_n \in [0, 1)$, $\lim_{n\to\infty} \frac{\tau_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\theta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$.

If $I - T$ and $I - S$ are demiclosed at 0*, the sequence* $\{x_n\}$ *converges strongly to* $\hat{x} = P_\Omega \xi(\hat{x}) \in \Omega$.

Remark 3.2 According to the existing results in Wang (2017) and Yao et al. (2018), the strong convergence results of both were obtained by employing the Halpern algorithm. If the Meir–Keeler contraction mapping is a constant mapping, that is, $\xi \equiv u$ (u is a constant), this also shows that the Meir–Keeler contraction algorithm is equivalent to the Halpern algorithm under special circumstances. Then, the following self-adaptive inertial Halpern algorithm for demicontractive mappings is obtained, and extends the existing results in Wang (2017) and Yao et al. (2018).

Corollary 3.4 *Let H_1 and H_2 be two Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*, $T : H_1 \to H_1$, and $S : H_2 \to H_2$ be demicontractive mappings with coefficients $k_1 \in (-\infty, 1)$ and $k_2 \in (-\infty, 1)$, respectively. For any initial points $x_0, x_1 \in H_1$, the iterative sequence $\{x_n\}$ is generated by the following algorithm:*

$$\begin{cases} w_n = x_n + \tau_n(x_n - x_{n-1}), \\ u_n = w_n - \beta_n \left[(I - T)w_n + A^*(I - S)Aw_n\right], \\ x_{n+1} = \theta_n \xi(u) + (1 - \theta_n)u_n, \ \forall n \geq 1, \end{cases} \tag{3.11}$$

where if $(I - S)Aw_n \neq 0$, the self-adaptive stepsize $\beta_n = \sigma_n \min\left\{\frac{1-k_1}{2}, \frac{(1-k_2)\|(I-S)Aw_n\|^2}{2\|A^*(I-S)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$; otherwise, $\beta_n = \sigma_n(1 - k_1)/2$. Furthermore, $\tau_n \in [0, 1)$, $\lim_{n\to\infty} \frac{\tau_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\theta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$.

If $I - T$ and $I - S$ are demiclosed at 0*, $F(T) \neq \emptyset$, and $F(S) \neq \emptyset$, the iterative sequence $\{x_n\}$ converges strongly to $\hat{x} = P_\Omega u \in \Omega$.*

4 Theoretical applications

4.1 Split feasibility problems

Let C and Q be two nonempty closed convex subsets of two Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator with the corresponding adjoint operator A^*. Let $P_C : H_1 \to C$ and $P_Q : H_2 \to Q$ be two metric projection operators, and $\xi : H_1 \to H_1$ be a Meir–Keeler contraction mapping. For any initial points $x_0, x_1 \in H_1$, the iterative sequence $\{x_n\}$ of the split feasibility problem is generated by the following algorithm:

$$\begin{cases} w_n = x_n + \tau_n(x_n - x_{n-1}), \\ u_n = w_n - \beta_n \left[(I - P_C)w_n + A^*(I - P_Q)Aw_n\right], \\ x_{n+1} = \theta_n\xi(u_n) + (1 - \theta_n)u_n, \ \forall n \geq 1, \end{cases} \tag{4.1}$$

where if $(I - P_Q)Aw_n \neq 0$, the self-adaptive stepsize $\beta_n = \sigma_n \min\left\{1, \frac{\|(I-P_Q)Aw_n\|^2}{\|A^*(I-P_Q)Aw_n\|^2}\right\}$ with $\sigma_n \in (0, 1)$; otherwise, $\beta_n = \sigma_n$. Furthermore, $\tau_n \in [0, 1)$, $\lim_{n\to\infty} \frac{\tau_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\theta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$.

Theorem 4.1 *If the solution set of split feasibility problem $\Psi = \{x^* : x^* \in C, Ax^* \in Q\} \neq \emptyset$, the iterative sequence $\{x_n\}$ generated by algorithm (4.1) converges strongly to $\hat{x} = P_\Psi\xi(\hat{x}) \in \Psi$.*

Proof From the definitions of the demicontractive mapping and the metric projection, we know that the metric projections P_C and P_Q are demicontractive mappings with coefficient $k = -1$, the fixed point sets of metric projection operators P_C and P_Q are C and Q, respectively. In addition, it is well known that the metric projection is demiclosed to 0. According to Theorem 3.1, the strong convergence results of $\{x_n\}$ generated by algorithm (4.1) are obtained. □

4.2 Split equilibrium problems

Let C be a nonempty closed and convex subset of a Hilbert space H. The well-known equilibrium problem is to find a point $z \in C$ satisfy $\mathcal{F}(z, x) \geq 0, \ \forall x \in C$, where $\mathcal{F} : C \times C \to R$ is a bifunction and the following conditions hold.

(E1) $\mathcal{F}(x, x) = 0, \forall x \in C$.
(E2) $\mathcal{F}(x, y) + \mathcal{F}(y, x) \leq 0, \forall x, y \in C$.
(E3) For any $x, y, z \in C$, $\limsup_{a\to 0^+} \mathcal{F}(az + (1 - a)x, y) \leq \mathcal{F}(x, y)$.
(E4) For each $x \in C$, the function $y \mapsto \mathcal{F}(x, y)$ is convex and lower semi-continuous.

The solution set of equilibrium problem is denoted by $EP(\mathcal{F})$. The equilibrium problem has the following important properties.

Lemma 4.1 (Combettes and Hirstoaga 2005) *Let C be a nonempty closed convex subset of a Hilbert space H, and let $\mathcal{F} : C \times C \to R$ be a bifunction satisfying* (E1)–(E4). *For each $r > 0$ and $x \in H$, there exists a point $z \in C$, such that:*

$$\mathcal{F}(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \ \forall y \in C.$$

Set $T_r^{\mathcal{F}}(x) = \{z \in C : \mathcal{F}(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \ \forall x \in H$. Then:

1. $T_r^{\mathcal{F}}$ *is single-valued.*
2. $T_r^{\mathcal{F}}$ *is firmly nonexpansive, i.e.,* $\|T_r^{\mathcal{F}}x - T_r^{\mathcal{F}}y\|^2 \leq \langle T_r^{\mathcal{F}}x - T_r^{\mathcal{F}}y, x-y\rangle$.
3. $F(T_r^{\mathcal{F}}) = EP(\mathcal{F})$ *is nonempty, closed, and convex.*

Definition 4.1 Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ be a bounded linear operator, $\mathcal{F} : C \times C \to R$ and $\mathcal{K} : Q \times Q \to R$ be two bifunctions satisfying (E1)–(E4). The split equilibrium problem is to find:

$$x^* \in EP(\mathcal{F}), \quad \text{and} \quad Ax^* \in EP(\mathcal{K}).$$

Let C and Q be two nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, $\mathcal{F} : C \times C \to R$ and $\mathcal{K} : Q \times Q \to R$ be bifunctions satisfying (E1)–(E4), let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*, and $\xi : H_1 \to H_1$ be a Meir–Keeler contraction mapping. For any initial points $x_0, x_1 \in H_1$, the iterative sequence $\{x_n\}$ of the split equilibrium problem is generated by the following algorithm:

$$\begin{cases} w_n = x_n + \tau_n(x_n - x_{n-1}), \\ u_n = w_n - \beta_n\left[(I - T_r^{\mathcal{F}})w_n + A^*(I - T_r^{\mathcal{K}})Aw_n\right], \\ x_{n+1} = \theta_n\xi(u_n) + (1-\theta_n)u_n, \quad \forall n \geq 1, \end{cases} \tag{4.2}$$

where if $(I - T_r^{\mathcal{K}})Aw_n \neq 0$, the self-adaptive stepsize $\beta_n = \sigma_n \min\left\{1, \frac{\|(I-T_r^{\mathcal{K}})Aw_n\|^2}{\|A^*(I-T_r^{\mathcal{K}})Aw_n\|^2}\right\}$ with $\sigma_n \in (0,1)$; otherwise, $\beta_n = \sigma_n$. Furthermore, $\tau_n \in [0,1)$, $\lim_{n\to\infty}\frac{\tau_n}{\theta_n}\|x_n - x_{n-1}\| = 0$, $\theta_n \in (0,1)$ with $\sum_{n=1}^{\infty}\theta_n = \infty$ and $\theta_n \to 0$ as $n \to \infty$.

Theorem 4.2 *If* $\Gamma = \{x^* \in H : x^* \in EP(\mathcal{F}),\ Ax^* \in EP(\mathcal{K})\} \neq \emptyset$, *the iterative sequence* $\{x_n\}$ *generated by algorithm* (4.2) *converges strongly to* $\hat{x} = P_\Gamma\xi(\hat{x}) \in \Gamma$.

Proof From the definitions of the demicontractive mapping and the firmly nonexpansive, when the fixed point set is nonempty, the firmly nonexpansive mapping is the demicontractive mapping with coefficient $k = -1$. In addition, it is well known that the firmly nonexpansive is demiclosed to 0. According to Theorem 3.1 and Lemma 4.1, the results of strong convergence of sequence $\{x_n\}$ generated by algorithm (4.2) are obtained. □

5 Numerical examples

In this section, we provide some numerical examples to demonstrate the effectiveness and realization of convergence behavior of Theorem 3.1. All codes were written in Matlab R2018b, and ran on a Lenovo ideapad 720S with 1.6 GHz Intel Core i5 processor and 8GB of RAM. Our results compare the existing conclusions below. First, we give these theorems and iterative algorithms as follows.

Theorem 5.1 (Censor and Segal 2009) *Let* $H_1 = R^N$ *and* $H_2 = R^M$, *A be a matrix* $R^{M\times N}$. *Let* $T : H_1 \to H_1$ *and* $S : H_2 \to H_2$ *be directed mappings. The iterative sequence* $\{x_n\}$ *of the split common fixed point problem* (1.1) *is generated by the following iterative scheme:*

$$x_{n+1} = T(x_n - \beta A^*(I - S)Ax_n), \quad \forall n \geq 1, \tag{5.1}$$

where A^{T} *is the matrix transposition of A, M is the largest eigenvalue of matrix* $A^{\mathrm{T}}A$, *and* $\beta \in (0, 2/M)$. *If* $I - T$ *and* $I - S$ *are demiclosed at* 0, $F(T) \neq \emptyset$ *and* $F(S) \neq \emptyset$, *the iterative sequence* $\{x_n\}$ *converges to a point* $\hat{x} \in \Omega$.

Theorem 5.2 (Moudafi 2010) *Let H_1 and H_2 be Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be demicontractive mappings with coefficients $k_1 \in [0, 1)$ and $k_2 \in [0, 1)$, respectively. The iterative sequence $\{x_n\}$ of the split common fixed point problem (1.1) generated by the following iterative scheme:*

$$\begin{cases} u_n = x_n - \beta A^*(I - S)Ax_n, \\ x_{n+1} = (1 - \delta_n)u_n + \delta_n T u_n, \quad \forall n \geq 1, \end{cases} \tag{5.2}$$

where M is the spectral radius of matrix A^*A, $\beta \in (0, (1 - k_2)/M)$ and $\delta_n \in (\epsilon, 1 - k_1 - \epsilon)$ for a small enough $\epsilon > 0$. If $I - T$ and $I - S$ are demiclosed at 0, $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, the iterative sequence $\{x_n\}$ converges weakly to a point $\hat{x} \in \Omega$.

Theorem 5.3 (Boikanyo 2015) *Let H_1 and H_2 be Hilbert spaces, and $A : H_1 \to H_2$ be a bounded linear operator with the adjoint operator A^*. Let $T : H_1 \to H_1$ be a directed mapping and $S : H_2 \to H_2$ be a demicontractive mapping with coefficient $k_2 \in (-\infty, 1)$, respectively. The iterative sequence $\{x_n\}$ of the split common fixed point problem (1.1) is generated by the following iterative scheme:*

$$\begin{cases} u_n = x_n - \beta_n A^*(I - S)Ax_n, \\ x_{n+1} = \delta_n u + (1 - \delta_n)((1 - \omega)u_n + \omega T u_n), \quad \forall n \geq 1, \end{cases} \tag{5.3}$$

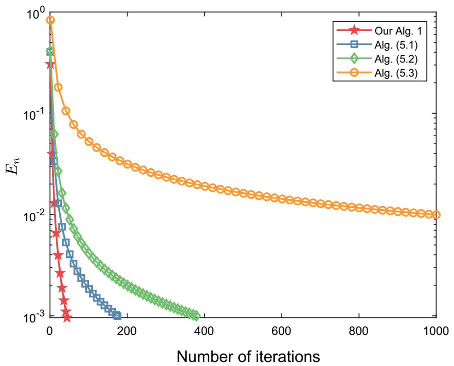
where $\beta_n = \frac{(1-k_2)\|(I-S)Ax_n\|^2}{2\|A^*(I-S)Ax_n\|^2}$ with $Ax_n \neq SAx_n$; otherwise, $\beta_n = 0$. $\omega \in (0, 1 - k_1)$ and $\delta_n \in (0, 1)$ with $\sum_{n=1}^{\infty} \delta_n = \infty$ and $\delta_n \to 0$ as $n \to \infty$. If $I - T$ and $I - S$ are demiclosed at 0, $F(T) \neq \emptyset$ and $F(S) \neq \emptyset$, the iterative sequence $\{x_n\}$ converges strongly to a point $\hat{x} \in \Omega$.

Example 5.1 Set $C = \{(x_1, x_2, x_3) \in H_1 : x_2^2 + x_3^2 - 1 \leq 0\}$, $Q = \{(y_1, y_2, y_3) \in H_2 : y_1^2 - y_2 + 5 \leq 0\}$, $T = P_C$ and $S = P_Q$, in addition, $A = \begin{bmatrix} \sqrt{5} & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. It is easy to check that $x^* = (0, 1, 0)$ is a unique solution of split common fixed point problem (1.1).

Our parameters are set as follows. In our Algorithm 1, set $\rho_n = \frac{1}{(n+1)^2}$, $\tau = 0.5$, $\sigma_n = 0.5$, $\theta_n = \frac{1}{n+1}$ and $\xi(u_n) = 0.3u_n$. In Algorithm (5.1), set $\beta = \frac{1.8}{\|A\|^2}$. In Algorithm (5.2), set $\beta = \frac{1.8}{\|A\|^2}$, $\delta_n = 0.6$. In Algorithm (5.3), set $\delta_n = \frac{1}{n+1}$, $u = x_0$ and $\omega = 0.5$. The error of the iterative algorithms is denoted by $E_n = \|x_n - x^*\|^2$. Take initial points x_0, x_1 which are generated randomly in MATLAB and $E_n < 10^{-3}$ or maximum iteration 1000 as the stopping criterion. Our numerical results are shown in Fig. 1.

Example 5.2 In this example, we consider $H = L_2([0, 2\pi])$ with the inner product $\langle x, y \rangle := \int_0^{2\pi} x(t)y(t)\mathrm{d}t$ and with the associated norm which given by $\|x\|_2 := \left(\int_0^{2\pi} |x(t)|^2\mathrm{d}t\right)^{\frac{1}{2}}$, $\forall x, y \in L_2([0, 2\pi])$. We also consider the following half-space:

$$C = \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} x(t)\mathrm{d}t \leq 1\right\} \text{ and}$$
$$Q = \left\{x \in L_2([0, 2\pi]) : \int_0^{2\pi} |x(t) - \sin(t)|^2\,\mathrm{d}t \leq 16\right\}.$$



(a) Case I **(b)** Case II

Fig. 1 Numerical results for Example 5.1

Defining a linear continuous operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$, where $(Ax)(t) := x(t)$. Then, $(A^*x)(t) = x(t)$ and $\|A\| = 1$. Now, we solve the split common fixed point problem: find $x^* \in F(T)$, such that $Ax^* \in F(S)$, where $T = P_C$ and $S = P_Q$. For our numerical computation, we can also write the projections onto set C and the projections onto set Q as follows:

$$P_C(z) = \begin{cases} \frac{1-\int_0^{2\pi} z(t)dt}{4\pi^2} + z, & \int_0^{2\pi} z(t)\mathrm{d}t > 1, \\ z, & \int_0^{2\pi} z(t)\mathrm{d}t \le 1, \end{cases}$$

and

$$P_Q(w) = \begin{cases} \sin + \frac{4}{\sqrt{\int_0^{2\pi} |w(t)-\sin(t)|^2 \mathrm{d}t}}(w - \sin), & \int_0^{2\pi} |w(t) - \sin(t)|^2 \mathrm{d}t > 16, \\ w, & \int_0^{2\pi} |w(t) - \sin(t)|^2 \mathrm{d}t \le 16. \end{cases}$$

We consider different initial values x_0 and x_1. The error of the iterative algorithms is denoted by:

$$E_n = \frac{1}{2}\|T(x_n) - x_n\|_2^2 + \frac{1}{2}\|S(A(x_n)) - A(x_n)\|_2^2.$$

Our parameter settings are the same as in Example 5.1. Take $E_n < 10^{-3}$ or maximum iteration 200 as the stopping criterion. Our numerical results are shown in Table 1 and Fig. 2. In Table 1, “Iter.” and “Time(s)” denote the number of iterations and the cpu time in seconds, respectively.

Example 5.3 We consider a linear inverse problem: $b = Ax_0 + w$, where $x_0 \in R^N$ is the (unknown) signal to recover, $w \in R^M$ is a noise vector, and $A \in R^{M\times N}$ models the acquisition device. To recover an approximation of the signal x_0, we use the Basis Pursuit denoising method, that is, use the ℓ_1 norm as a sparsity enforcing penalty:

$$\min_{x\in R^N} \Phi(x) = \frac{1}{2}\|b - Ax\|^2 + \lambda\|x\|_1, \tag{5.4}$$

Table 1 Numerical results for Example 5.2

Cases	Initial values	Our Alg. 1		Alg. (5.1)		Alg. (5.2)		Alg. (5.3)	
		Iter.	Time(s)	Iter.	Time(s)	Iter.	Time(s)	Iter.	Time(s)
I	$x_0 = 2t^2, x_1 = \frac{t^3}{10}$	9	3.3311	10	2.4528	17	4.1646	200	50.5499
II	$x_0 = t^2, x_1 = 2^t$	11	3.8928	14	3.5019	20	5.3951	200	55.4820
III	$x_0 = t^2, x_1 = e^t$	11	3.9093	16	3.9798	19	4.7086	200	50.3240
IV	$x_0 = \sin(t), x_1 = 3t^2$	13	4.4807	17	4.1929	21	5.2177	200	50.3049

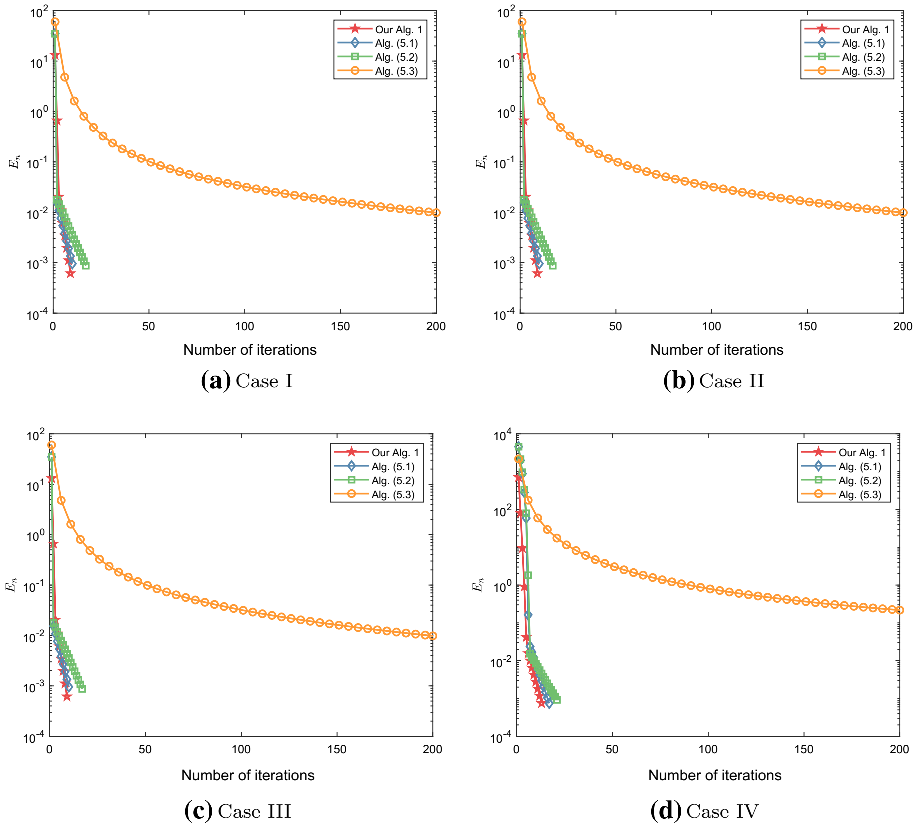


Fig. 2 Convergence behavior of iteration error $\{E_n\}$ with different initial values for Example 5.2

where $\|x\|_1 = \sum_i |x_i|$ and λ is a regularization parameter that closely relate to noise w. Using the idea of convex analysis, a minimizer of (5.4) is a solution of the constrained least-squares problem:

$$\min_{x \in R^N} \frac{1}{2}\|b - Ax\|_2^2 \text{ such that } \|x\|_1 < t, \tag{5.5}$$

for any non-negative real number t. Note that, our Algorithm 1 can be applied to approximate solutions of the problem (5.5), because it is a special case of the SCFPP, where

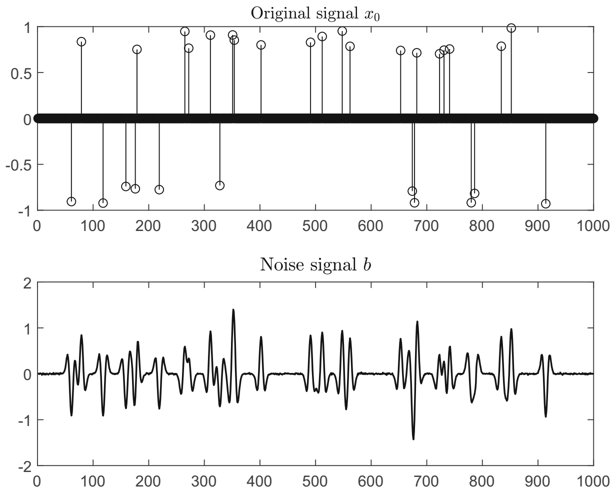


Fig. 3 Original and noisy signals for Example 5.3

$C = \left\{x \in R^N : \|x\|_1 < t\right\}$, $Q = \{b\}$, $T = P_C$, and $S = P_Q$. For more discussion, see López et al. (2012).

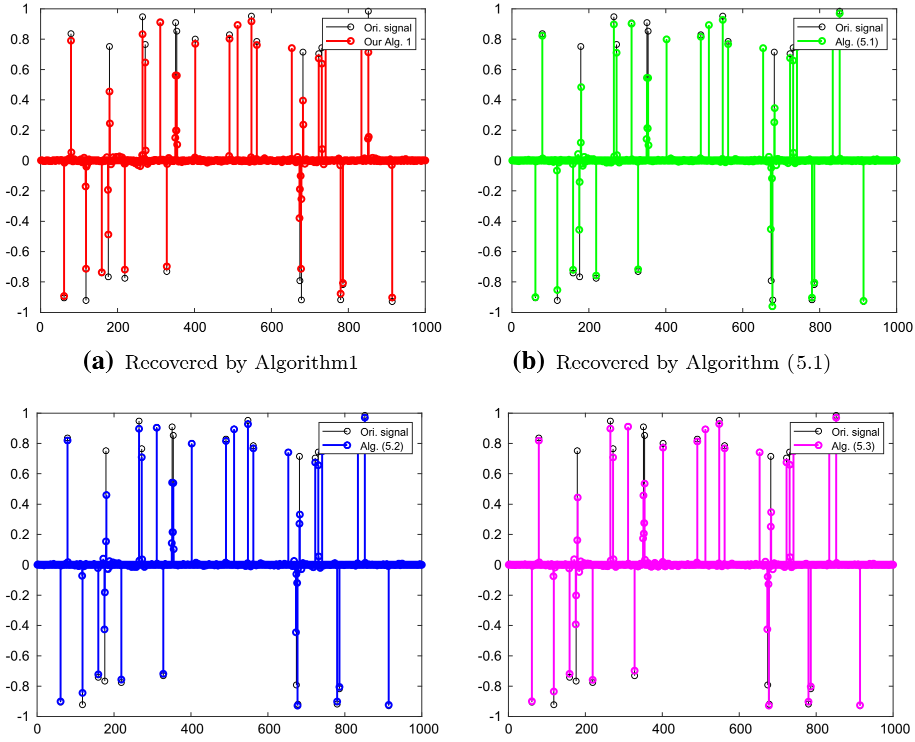
In our experiment, we want to recovery a sparse signal $x_0 \in R^N$ with k ($k \ll N$) non-zero elements. A simple linearized model of signal processing is to consider a linear operator, that is, a filtering $Ax = \varphi(x)$, where φ is a second derivative of Gaussian. We wish to solve $b = Ax_0 + w$, where w is a realization of Gaussian white noise with variance 10^{-2}. Hence, we need to solve the problem (5.5). Set $N = 1000$, $k = 30$, and $t = \|x_0\|_1$. Our parameter settings are the same as in Example 5.1. We take the maximum number of iterations 5×10^4 as a common stopping criterion. In addition, we use the signal-to-noise ratio (SNR) to measure the quality of recovery and a larger SNR means a better recovery quality. Numerical results are reported in Fig. 3 and Fig. 4. The SNR of Algorithm 1, Algorithm (5.1), Algorithm (5.2), and Algorithm (5.3) are 9.2100, 6.6486, 6.9471, and 6.9033, respectively.

Remark 5.1 (i) From Example 5.1–Example 5.3, we know that our proposed Algorithm 1 is faster than the Algorithm (5.1), the Algorithm (5.2), and the Algorithm (5.3). Observing the same expected results, our scheme is better.

(ii) In the case that the selection of the initial point does not affect the computing performance, our proposed algorithm is robust. (see Figs. 1 and 2).

6 Conclusion

The first conclusion from Sect. 3 is that we gave self-adaptive inertial Meir–Keeler contraction algorithms to approximate the solution of the split common fixed point problem (1.1) in the framework of infinite Hilbert spaces. It is worth noting that the corresponding strong convergence theorems are obtained without prior knowledge of operator norms. An important distinction between our Algorithm 1 and the existing results is that the excellent stability and better convergence rate of our algorithm are guaranteed by the proposed self-adaptive stepsize sequence. In numerical examples, some numerical experiments and a signal recovery problem are performed to demonstrate the validity and authenticity of our algorithm. Furthermore,



(a) Recovered by Algorithm1 **(b)** Recovered by Algorithm (5.1)

(c) Recovered by Algorithm (5.2) **(d)** Recovered by Algorithm (5.3)

Fig. 4 Numerical results for Example 5.3

we compared with the existing results in Boikanyo (2015), Censor and Segal (2009) and Moudafi (2010), which implies that our Algorithm 1 is superior and stable.

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