Numerical Algorithms (2024) 95:117–147
https://doi.org/10.1007/s11075-023-01566-1

ORIGINAL PAPER

A modified generalized version of projected reflected gradient method in Hilbert spaces

Xiaolin Zhou[1] · Gang Cai[1] · Bing Tan[2] · Qiao-Li Dong[3]

Received: 5 December 2022 / Accepted: 22 April 2023 / Published online: 27 June 2023

Abstract

This article is concerned with a universal version of projected reflected gradient method with new step size for solving variational inequality problem in Hilbert spaces. Under appropriate assumptions controlled by the operators and parameters, we acquire the weak convergence of the proposed algorithm. Moreover, we establish an R-linear convergence rate of our method on the condition that the relevant mapping is strongly monotone. We rework our first algorithm so that it can be simplified to several generalized methods in the literature. The efficacy and availability of our proposed iterative scheme are demonstrated in numerical experiments.

Keywords Projected reflected gradient method · Variational inequality · Weak and linear convergence · Hilbert spaces

Mathematics Subject Classification 47H05 · 47H07 · 47H10 · 54H25

✉ Gang Cai
caigang-aaaa@163.com

Xiaolin Zhou
zhouxiaolinmath@163.com

Bing Tan
bingtan72@gmail.com

Qiao-Li Dong
dongql@lsec.cc.ac.cn

1 School of Mathematical Sciences, Chongqing Normal University, Chongqing 401331, China

2 School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China

3 Tianjin Key Laboratory for Advanced Signal Processing and College of Science, Civil Aviation University of China, Tianjin 300300, China

1 Introduction

Let's assume that C is a nonempty, closed and convex subset of a real Hilbert space H. Then, we think about the variational inequality problem (VIP, for short) as below: Find $u^* \in C$ with

$$\langle Au^*, u - u^* \rangle \geq 0, \quad \forall u \in C, \tag{1}$$

where $A : C \to H$ is a given mapping and it is continuous. The solution set of (1) is expressed as $S := VI(C, A)$ and we assume $S \neq \emptyset$.

VIP can be used as a fundamental tool in a broad range of optimization problems, economics and other associated problems (see, for example, [1–5]), so it has attracted the attention of many researchers. Therefore, many reseachers have come up with a great deal of different iterative methods for solving VIP (see, for example, [13, 15, 16, 23, 25, 28–33, 35]).

One of the researchers, Korpelevich, put forward the extragradient method in [6], whose iterative process is the following form:

$$\begin{cases} v_n = P_C\left(u_n - \mu A u_n\right), \\ u_{n+1} = P_C\left(u_n - \mu A v_n\right), \quad n \geq 0, \end{cases} \tag{2}$$

where $u_0 \in C$, $\mu \in (0, \frac{1}{L})$ and $L > 0$. We have to estimate the operator A twice and do two projections on C in each iteration. Sometimes, if C is a complex set, two projections will seriously affect the effectiveness and usage of the method.

To decrease the calculation cost of (2), many authors have tried to reduce the number of projections. Censor et al. [20] came up with the subgradient extragradient method and the iterative process of the new method is obtained by:

$$\begin{cases} v_n = P_C\left(u_n - \mu A u_n\right), \\ T_n := \{w \in H : \langle u_n - \mu A u_n - v_n, w - v_n \rangle \leq 0\}, \\ u_{n+1} = P_{T_n}\left(u_n - \mu A v_n\right), \quad n \geq 0, \end{cases} \tag{3}$$

where $u_0 \in H$, $\mu \in (0, \frac{1}{L})$ and $L > 0$. Apparently, (3) still needs to calculate the operator A twice in each iteration. The second projection of (3) is projected into the half space T_n, which possesses an explicit formula. Thus, the subgradient extragradient method is better than the extragradient method.

Later, Malitsky and Semenov referred to the ideas of algorithms in [20, 21] and showed an effective and improved algorithm in [22]:

$$\begin{cases} T_n := \{w \in H : \langle u_n - \mu A v_{n-1} - v_n, w - v_n \rangle \leq 0\}, \\ u_{n+1} = P_{T_n}\left(u_n - \mu A v_n\right), \\ v_{n+1} = P_C\left(u_{n+1} - \mu A v_n\right), \quad n \geq 0, \end{cases} \tag{4}$$

where $\mu \in (0, \frac{1}{3L}]$. This method only computes one projection onto the admissible set per one iteration.

In 2015, the projected reflected gradient method was discovered by Malitsky in [7] and its iterative process as below:

$$\begin{cases} u_{n+1} = P_C\left(u_n - \mu_n A v_n\right), \\ v_{n+1} = \left(2u_{n+1} - v_n\right), \quad n \geq 0, \end{cases} \tag{5}$$

where $\mu_n \in (0, \frac{\sqrt{2}-1}{L})$ and $L > 0$. In each iteration, (5) is same with (4) regarding the number of projections onto the admissible set. However, this method holds a simpler form. Many authors have proposed a great deal of improved schemes (see, for example, [8, 18, 24, 26, 27]), which are based on the projected reflected gradient method.

Dong et al. [9] introduced a general inertial projected gradient method to find a solution for VIP, its form is that: $u_0 \in C$ and $v_0, \omega_0 \in H$,

$$\begin{cases} u_{n+1} = P_C\left(\omega_n - \mu_n A v_n\right), \\ v_{n+1} = u_{n+1} + \delta\left(u_{n+1} - u_n\right), \\ \omega_{n+1} = u_{n+1} + \theta\left(u_{n+1} - u_n\right), \quad n \geq 0, \end{cases} \tag{6}$$

where $\delta \in (1, \infty)$ and $\theta \in [0, \frac{\delta(\delta-1)}{3\delta^2-1})$. The differences between this method and above methods are that it takes an adaptive step size and uses the inertial extrapolation step.

Olaniyi and Yekini combined the characteristics of the above algorithms and developed a generalized projected reflected gradient method in [11]: $u_0 \in C$ and $v_0, \omega_0 \in H$,

$$\begin{cases} u_{n+1} = P_C\left(\omega_n - \mu_n A v_n\right), \\ v_{n+1} = u_{n+1} + \delta\left(u_{n+1} - u_n\right), \\ \omega_{n+1} = u_{n+1} + \theta\left(u_{n+1} - u_n\right), \quad n \geq 0, \end{cases} \tag{7}$$

where $\delta > 0, 0 \leq \theta < \frac{\delta}{2\delta+1}$ and $2\delta^2(1-3\theta) > \delta - \theta + 2\delta\theta$. We notice that this method selects two inertial parameters and only computes one projection and an operator in each iteration.

Based on the understanding of above algorithms, we innovate a new projected gradient method for the sake of solving VIP. We testify the weak and linear convergence of our algorithm in detail. The biggest highlight of our algorithm is that it combines a variable step size, which is controlled by multiple parameters δ, θ and ε. Moreover, we improve our algorithm into a new one, which can be simplified to some classical methods in the literature. The improvements of our article are not only reflected in the parameters, but also in the definition of step size. It is apparent to observe that we can make numerous accelerated improvements and repeatedly optimize our algorithm by our ideas. The data experiments show that if different parameters are adopted, the convergence rate of our algorithm can be improved in different degrees.

The rest of our article is structured as below: In Sect. 2, we review some definitions and lemmas to be adopted in our convergence analysis. Our own algorithm is on display in Sect. 3. In Sect. 4, the weak and linear convergence of our algorithm are verified in detail. In Sect. 5, we innovate a modified algorithm with an improved step size. Section 6 mainly includes our numerical experiments and Sect. 7 expresses the conclusions.

2 Preliminaries

In this section, we will state some basic definitions and lemmas used in the text.

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of $\{u_n\}_{n=1}^{\infty}$ to u is denoted by $u_n \rightharpoonup u$ as $n \to \infty$, while the strong convergence of $\{u_n\}_{n=1}^{\infty}$ to u is denoted by $u_n \to u$ as $n \to \infty$.

Definition 1 If C is a nonempty, closed and convex subset of H. In that way, P_C is called the metric projection of H onto C, if for any $x \in H$, we have a unique $P_C x \in C$ such that

$$\|x - P_C x\| \le \|x - v\|, \quad \forall v \in C.$$

It is easy to see that P_C is a firmly nonexpansive mapping of H onto C, that is,

$$\langle u - v, P_C u - P_C v \rangle \ge \|P_C u - P_C v\|^2, \quad \forall u, v \in H.$$

Moreover, we have

$$\langle u - P_C u, P_C u - v \rangle \ge 0, \quad \forall v \in C,$$

which implies that

$$\|u - v\|^2 \ge \|u - P_C u\|^2 + \|v - P_C u\|^2, \quad \forall u \in H, v \in C.$$

Definition 2 If $A : H \to H$ is an operator,

(i) A is said to be L-Lipschitz continuous if there exists some $L > 0$ such that

$$\|Au - Av\| \le L\|u - v\|, \quad \forall u, v \in H.$$

(ii) A is said to be γ-strongly monotone if there exists some $\gamma > 0$ such that

$$\langle Au - Av, u - v \rangle \ge \gamma\|u - v\|^2, \quad \forall u, v \in H.$$

(iii) A is said to be monotone, if

$$\langle Au - Av, u - v \rangle \ge 0, \quad \forall u, v \in H.$$

Lemma 1 *([12]) The following two equalities are common in H:*

(i) $2\langle u, v \rangle = \|u\|^2 + \|v\|^2 - \|u - v\|^2 = \|u + v\|^2 - \|u\|^2 - \|v\|^2, \quad \forall u, v \in H;$

(ii) $\|\alpha u + (1-\alpha)v\|^2 = \alpha\|u\|^2 + (1-\alpha)\|v\|^2 - \alpha(1-\alpha)\|u - v\|^2, \quad \forall u, v \in H$ *and* $\alpha \in \mathbb{R}$.

Lemma 2 *([19]) Given a mapping $A : C \to H$ and $z \in C$, if A is continuous and monotone. Then $z \in S \Leftrightarrow \langle Au, u - z \rangle \ge 0$, $\forall u \in C$.*

Lemma 3 *([7]) Suppose that $\{s_n\}$ and $\{t_n\}$ are two real sequences, for any $n \ge 0$, $s_n \ge 0$ and $t_n \ge 0$. If we have the following inequality:*

$$s_{n+1} \le s_n - t_n,$$

then we get that $\{s_n\}$ *is bounded and* $\lim_{n\to\infty} t_n = 0$.

Lemma 4 *([18]) Let* $\{s_n\}$ *and* $\{t_n\}$ *be two sequences of nonnegative numbers such that*

$$s_{n+1} + t_{n+1} \le (1-\rho)s_n + r\rho s_{n-1} + r t_n,$$

where $\rho \in (0, +\infty)$ *and* $r \in (0, 1)$. *Then the sequence* $\{s_n\}$ *converges linearly to* 0.

3 Proposed algorithm

Now we introduce our first algorithm for solving VIP, which is derived from the projected gradient method in [11]. In order to precisely obtain the weak and linear convergence results of our method, we need the following conditions:

(C1) C is a nonempty, closed and convex subset of H.
(C2) $A : H \to H$ is monotone and L-Lipschitz continuous.
(C3) The solution S for VIP is nonempty.
(C4) $\delta > 1, \varepsilon > 0$.
(C5) $\theta \in [0, \infty)$, $0 \le \theta < \frac{\delta}{2\delta+1}$ and $2\delta^2(1-3\theta) > \delta - \theta + 2\delta\theta$.

We provide a new algorithm as follows:

Algorithm 1 Projected reflected gradient method with adaptive step size.

Initialization: Let $\mu_0 > 0$ and set $n := 0$. Assume

$$\overline{\eta} = \min\{\frac{\delta-(1+2\delta)\theta}{2\delta^2(\varepsilon^2+\varepsilon+1)}, \frac{(\delta-\theta)}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}, \frac{2\delta^2(1-3\theta)-\delta+\theta-2\delta\theta}{2\delta^2(\varepsilon^2+\varepsilon+1)}, \frac{2\delta^2(1-3\theta)-\delta+\theta}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}\},$$

choose $\eta \in (0, \overline{\eta})$ and let $u_0 \in C,\ v_0, \omega_0 \in H$ be any three starting points.
Iterative Steps: u_{n+1} is determined via the previous v_n and ω_n as follows:
Step 1. Compute

$$\begin{cases} u_{n+1} = P_C(\omega_n - \mu_n A v_n), \\ v_{n+1} = u_{n+1} + \delta(u_{n+1} - u_n), \\ \omega_{n+1} = u_{n+1} + \theta(u_{n+1} - u_n). \end{cases}$$

If we have $\omega_n = v_n = u_{n+1}$, then Stop. Conversely, go to Step 2.
Step 2. Let

$$\rho_n := (\varepsilon+1)\|v_{n-1} - u_n\|^2 + \frac{\varepsilon+1}{\varepsilon}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2,$$

and turn the original step size into a new

$$\mu_{n+1} = \begin{cases} \mu_n, & \text{if} \langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle \le 0, \\ \min\{\frac{\eta\rho_n}{\langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle}, \mu_n\}, & \text{otherwise}. \end{cases} \tag{8}$$

Step 3. Next: set $n := n + 1$ and go to **Step 1.**

Remark 1 We see that $\{\mu_n\}$ is a decreasing sequence, which holds a positive lower bound:

$$\mu_n \geq \min\{\mu_0, \frac{2\varepsilon\eta}{L}\} > 0, \quad \forall n \geq 0.$$

It is easy to see that $\mu_{n+1} \leq \mu_n, \forall n \geq 0$. Observe that

$$\begin{aligned}
&\|v_n - v_{n-1}\|^2 \\
&= \|v_n - u_n\|^2 + \|u_n - v_{n-1}\|^2 + 2\langle v_n - u_n, u_n - v_{n-1}\rangle \\
&\leq \|v_n - u_n\|^2 + \|u_n - v_{n-1}\|^2 + 2\|v_n - u_n\|\|u_n - v_{n-1}\| \\
&\leq \|v_n - u_n\|^2 + \|u_n - v_{n-1}\|^2 + \frac{1}{\varepsilon}\|v_n - u_n\|^2 + \varepsilon\|u_n - v_{n-1}\|^2 \\
&\leq \frac{\varepsilon + 1}{\varepsilon}\|v_n - u_n\|^2 + (\varepsilon + 1)\|u_n - v_{n-1}\|^2.
\end{aligned}$$

From the definition of ρ_n, when $\langle Av_{n-1} - Av_n, u_{n+1} - v_n\rangle > 0$, we know that

$$\begin{aligned}
&\frac{\eta\rho_n}{\langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle} \\
&\geq \frac{\eta(\|v_n - v_{n-1}\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2)}{\langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle} \\
&\geq \frac{2\varepsilon\eta\|v_n - v_{n-1}\|\|u_{n+1} - v_n\|}{\langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle} \\
&\geq \frac{2\varepsilon\eta}{L},
\end{aligned}$$

which implies

$$\mu_n \geq \min\{\mu_0, \frac{2\varepsilon\eta}{L}\}, \quad \forall n \geq 0.$$

So, there exists $\mu > 0$ such that

$$\lim_{n\to\infty} \mu_n = \mu \geq \min\{\mu_0, \frac{2\varepsilon\eta}{L}\} > 0.$$

4 Convergence results

This section considers the weak convergence of Algorithm 1. Meanwhile, the linear convergence rate of the Algorithm 1 is shown in detail.

4.1 Weak convergence

First, we establish a lemma that plays a crucial role in our discussion of convergence.

Lemma 5 *Suppose that the conditions* $(C1)$-$(C5)$ *are satisfied and* $\{u_n\}$ *is a sequence generated by Algorithm 1. For any* $u^* \in S$*, let*

$$s_n = \|u_n - u^*\|^2 - \theta\|u_{n-1} - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2 \\ + [\frac{\theta\mu_n}{\delta\mu_{n-1}} + 2\eta(\varepsilon + 1)\frac{\mu_n}{\mu_{n+1}}]\|u_n - v_{n-1}\|^2 + 2\mu_{n-1}\delta\langle Au^*, u_{n-1} - u^*\rangle,$$

and

$$t_n = [1 - 3\theta - \frac{(\delta - \theta)\mu_n}{\delta^2\mu_{n-1}} + \frac{\mu_n}{2\mu_{n-1}\delta} - \frac{\beta_n}{2}]\|u_{n+1} - u_n\|^2,$$

where

$$\beta_n = \max\{\frac{(\delta + 1)\theta\mu_n}{\delta^2\mu_{n-1}} + 2\eta\varepsilon^2\frac{\mu_n}{\mu_{n+1}} + \frac{\theta\mu_{n+1}}{\delta\mu_n} + 2\eta(\varepsilon + 1)\frac{\mu_{n+1}}{\mu_{n+2}}, \\ \frac{\theta\mu_n}{\delta^2\mu_{n-1}} + 2\eta\frac{\varepsilon + 1}{\varepsilon}\frac{\mu_n}{\mu_{n+1}}\}.$$

Then there exists some $n_0 \in \mathbb{N}$ *such that* $s_{n+1} \le s_n - t_n$ *and* $t_n \ge 0$ *for any* $n \ge n_0$.

Proof Since $u_n = P_C(w_{n-1} - \mu_{n-1}Av_{n-1})$, we have

$$\langle u_n - \omega_{n-1} + \mu_{n-1}Av_{n-1}, u - u_n\rangle \ge 0, \quad \forall u \in C. \tag{9}$$

First let $u = u_{n+1}$ and substitute it into (9), we obtain

$$\langle u_n - \omega_{n-1} + \mu_{n-1}Av_{n-1}, u_{n+1} - u_n\rangle \ge 0, \tag{10}$$

then let $u = u_{n-1}$ and substitute it into (9), we have

$$\langle u_n - \omega_{n-1} + \mu_{n-1}Av_{n-1}, u_{n-1} - u_n\rangle \ge 0. \tag{11}$$

Substituting δ times of (11) into (10), we get

$$\langle u_n - \omega_{n-1} + \mu_{n-1}Av_{n-1}, u_{n+1} - u_n + \delta(u_{n-1} - u_n)\rangle \ge 0. \tag{12}$$

Owing to $v_n = u_n + \delta(u_n - u_{n-1})$, we get

$$\langle u_n - \omega_{n-1} + \mu_{n-1}Av_{n-1}, u_{n+1} - v_n\rangle \ge 0. \tag{13}$$

Equivalently, we get

$$\langle \mu_{n-1}Av_{n-1}, v_n - u_{n+1}\rangle \le \langle u_n - \omega_{n-1}, u_{n+1} - v_n\rangle. \tag{14}$$

By $\omega_n = u_n + \theta(u_n - u_{n-1})$, we obtain

$$\begin{aligned} u_n - \omega_{n-1} &= u_n - u_{n-1} - \theta(u_{n-1} - u_{n-2}) \\ &= u_n - u_{n-1} - \frac{\theta}{\delta}[(u_n - u_{n-1}) - (u_n - v_{n-1})] \\ &= \frac{\delta - \theta}{\delta}(u_n - u_{n-1}) + \frac{\theta}{\delta}(u_n - v_{n-1}) \\ &= \frac{\delta - \theta}{\delta^2}(v_n - u_n) + \frac{\theta}{\delta}(u_n - v_{n-1}). \end{aligned} \tag{15}$$

Observe that

$$2\langle v_n - u_n, u_{n+1} - v_n\rangle = \|u_{n+1} - u_n\|^2 - \|u_n - v_n\|^2 - \|u_{n+1} - v_n\|^2,$$

and

$$\begin{aligned} &2\langle u_n - v_{n-1}, u_{n+1} - v_n\rangle \\ &= \|u_n - v_{n-1}\|^2 + \|u_{n+1} - v_n\|^2 - \|u_n - v_{n-1} - u_{n+1} + v_n\|^2 \\ &\le \|u_n - v_{n-1}\|^2 + \|u_{n+1} - v_n\|^2. \end{aligned}$$

Substituting (15) into (14), we get

$$\begin{aligned} &2\mu_n\langle Av_{n-1}, v_n - u_{n+1}\rangle \\ &\le \frac{2\mu_n}{\mu_{n-1}}\langle u_n - \omega_{n-1}, u_{n+1} - v_n\rangle \\ &= \frac{2\mu_n(\delta - \theta)}{\mu_{n-1}\delta^2}\langle v_n - u_n, u_{n+1} - v_n\rangle + \frac{2\mu_n\theta}{\mu_{n-1}\delta}\langle u_n - v_{n-1}, u_{n+1} - v_n\rangle \\ &\le \frac{\mu_n(\delta - \theta)}{\mu_{n-1}\delta^2}[\|u_{n+1} - u_n\|^2 - \|u_n - v_n\|^2 - \|u_{n+1} - v_n\|^2] \\ &\quad + \frac{\mu_n\theta}{\mu_{n-1}\delta}[\|u_n - v_{n-1}\|^2 + \|u_{n+1} - v_n\|^2] \\ &= \frac{(\delta - \theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1} - u_n\|^2 + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 \\ &\quad + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 + [\frac{(\delta + 1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2. \end{aligned} \tag{16}$$

From $u_{n+1} = P_C(w_n - \mu_n Av_n)$, we have that for any $u \in C$

$$\langle u_{n+1} - \omega_n + \mu_n Av_n, u - u_{n+1}\rangle \ge 0.$$

In particular, we have

$$2\langle u_{n+1} - \omega_n + \mu_n Av_n, u_{n+1} - u^*\rangle \le 0. \tag{17}$$

Combining (16) and (17), we get

$$
\begin{aligned}
&2\mu_n\langle Av_{n-1}, v_n - u_{n+1}\rangle + 2\langle u_{n+1} - \omega_n + \mu_n Av_n, u_{n+1} - u^*\rangle \\
&\leq \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1} - u_n\|^2 + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 \\
&\quad + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2,
\end{aligned} \tag{18}
$$

which is equal to

$$
\begin{aligned}
&2\mu_n\langle Av_{n-1} - Av_n, v_n - u_{n+1}\rangle \\
&\quad + 2\mu_n\langle Av_n, v_n - u^*\rangle + 2\langle u_{n+1} - \omega_n, u_{n+1} - u^*\rangle \\
&\leq \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1} - u_n\|^2 + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 \\
&\quad + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2.
\end{aligned} \tag{19}
$$

Also, we have

$$
\begin{aligned}
&2\langle u_{n+1} - \omega_n, u_{n+1} - u^*\rangle \\
&= \|u_{n+1} - \omega_n\|^2 + \|u_{n+1} - u^*\|^2 - \|\omega_n - u^*\|^2.
\end{aligned} \tag{20}
$$

So, we write (19) as

$$
\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&\leq \|\omega_n - u^*\|^2 - \|u_{n+1} - \omega_n\|^2 + 2\mu_n\langle Av_{n-1} - Av_n, u_{n+1} - v_n\rangle \\
&\quad - 2\mu_n\langle Av_n, v_n - u^*\rangle + \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1} - u_n\|^2 \\
&\quad + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 \\
&\quad + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2.
\end{aligned} \tag{21}
$$

From the definition of μ_n, we get

$$
\begin{aligned}
&2\mu_n\langle Av_{n-1} - Av_n, u_{n+1} - v_n\rangle \\
&= \frac{\mu_n}{\mu_{n+1}}2\mu_{n+1}\langle Av_{n-1} - Av_n, u_{n+1} - v_n\rangle \\
&\leq \frac{\mu_n}{\mu_{n+1}}2\eta[(\varepsilon+1)\|u_n - v_{n-1}\|^2 \\
&\quad + \frac{\varepsilon+1}{\varepsilon}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2].
\end{aligned} \tag{22}
$$

Since A is monotone, it is obvious that

$$2\mu_n\langle Av_n, v_n - u^*\rangle \geq 2\mu_n\langle Au^*, v_n - u^*\rangle. \tag{23}$$

Since

$$\begin{aligned} v_n - u^* &= u_n + \delta(u_n - u_{n-1}) - u^* \\ &= (1+\delta)(u_n - u^*) - \delta(u_{n-1} - u^*). \end{aligned} \tag{24}$$

Therefore, we get

$$\begin{aligned} &2\mu_n\langle Au^*, v_n - u^*\rangle \\ &= 2\mu_n\langle Au^*, (1+\delta)(u_n - u^*) - \delta(u_{n-1} - u^*)\rangle \\ &= 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle - 2\mu_n\delta\langle Au^*, u_{n-1} - u^*\rangle. \end{aligned} \tag{25}$$

By (23) and (25), we get

$$\begin{aligned} &2\mu_n\langle Av_n, v_n - u^*\rangle \\ &\geq 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle - 2\mu_n\delta\langle Au^*, u_{n-1} - u^*\rangle. \end{aligned} \tag{26}$$

From (21), (22) and (26), we have

$$\begin{aligned} &\|u_{n+1} - u^*\|^2 \\ &\leq \|\omega_n - u^*\|^2 - \|u_{n+1} - \omega_n\|^2 \\ &\quad + \frac{\mu_n}{\mu_{n+1}} 2\eta[(\varepsilon+1)\|u_n - v_{n-1}\|^2 + \frac{\varepsilon+1}{\varepsilon}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2] \\ &\quad - 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle + 2\mu_n\delta\langle Au^*, u_{n-1} - u^*\rangle \\ &\quad + \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1} - u_n\|^2 + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 \\ &\quad + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2. \end{aligned} \tag{27}$$

Since

$$\begin{aligned} &2\langle u_{n+1} - u_n, u_n - u_{n-1}\rangle \\ &= \|u_{n+1} - u_n\|^2 + \|u_n - u_{n-1}\|^2 - \|u_{n+1} + u_{n-1} - 2u_n\|^2 \\ &\leq \|u_{n+1} - u_n\|^2 + \|u_n - u_{n-1}\|^2, \end{aligned}$$

we get

$$\begin{aligned} \|u_{n+1} - \omega_n\|^2 &= \|(u_{n+1} - u_n) - \theta(u_n - u_{n-1})\|^2 \\ &= \|u_{n+1} - u_n\|^2 - 2\theta\langle u_{n+1} - u_n, u_n - u_{n-1}\rangle + \theta^2\|u_n - u_{n-1}\|^2 \\ &\geq (1-\theta)\|u_{n+1} - u_n\|^2 + \theta(\theta-1)\|u_n - u_{n-1}\|^2. \end{aligned} \tag{28}$$

From Lemma 1, we obtain that

$$\begin{aligned}\|\omega_n - u^*\|^2 &= \|(1+\theta)(u_n - u^*) - \theta(u_{n-1} - u^*)\|^2 \\ &= (1+\theta)\|u_n - u^*\|^2 - \theta\|u_{n-1} - u^*\|^2 + \theta(1+\theta)\|u_n - u_{n-1}\|^2.\end{aligned} \tag{29}$$

Putting (28) and (29) into (27), we attain

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 \\ &\leq (1+\theta)\|u_n - u^*\|^2 - \theta\|u_{n-1} - u^*\|^2 + \theta(1+\theta)\|u_n - u_{n-1}\|^2 \\ &\quad - (1-\theta)\|u_{n+1} - u_n\|^2 - \theta(\theta-1)\|u_n - u_{n-1}\|^2 \\ &\quad + \frac{\mu_n}{\mu_{n+1}} 2\eta[(\varepsilon+1)\|u_n - v_{n-1}\|^2 + \frac{\varepsilon+1}{\varepsilon}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2] \\ &\quad - 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle + 2\mu_n\delta\langle Au^*, u_{n-1} - u^*\rangle \\ &\quad + \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1} - u_n\|^2 + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 \\ &\quad + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2.\end{aligned} \tag{30}$$

Equivalently, we have

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 \\ &\leq \|u_n - u^*\|^2 + \theta[\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2] \\ &\quad + [\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}} + \theta - 1]\|u_{n+1} - u_n\|^2 + 2\theta\|u_n - u_{n-1}\|^2 \\ &\quad + \frac{\mu_n}{\mu_{n+1}} 2\eta[(\varepsilon+1)\|u_n - v_{n-1}\|^2 + \frac{\varepsilon+1}{\varepsilon}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2] \\ &\quad - 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle + 2\mu_n\delta\langle Au^*, u_{n-1} - u^*\rangle \\ &\quad + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 \\ &\quad + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2.\end{aligned} \tag{31}$$

Observe that $\mu_n \leq \mu_{n-1}$. At the same time, since $u^* \in S$, we know that $\langle Au^*, u_n - u^*\rangle \geq 0$ and $\langle Au^*, u_{n-1} - u^*\rangle \geq 0$. So we get that

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 \\
&\leq \|u_n - u^*\|^2 + \theta[\|u_n - u^*\|^2 - \|u_{n-1} - u^*\|^2] \\
&\quad + [\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}} + \theta - 1]\|u_{n+1} - u_n\|^2 + 2\theta\|u_n - u_{n-1}\|^2 \\
&\quad + \frac{\mu_n}{\mu_{n+1}} 2\eta[(\varepsilon+1)\|u_n - v_{n-1}\|^2 + \frac{\varepsilon+1}{\varepsilon}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2] \\
&\quad - 2\mu_n\delta\langle Au^*, u_n - u^*\rangle + 2\mu_{n-1}\delta\langle Au^*, u_{n-1} - u^*\rangle \\
&\quad + (\frac{\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}})\|u_n - v_n\|^2 + \frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n - v_{n-1}\|^2 \\
&\quad + [\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - \frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1} - v_n\|^2.
\end{aligned} \tag{32}$$

Therefore from (32), we have

$$\begin{aligned}
&\|u_{n+1} - u^*\|^2 - \theta\|u_n - u^*\|^2 + 2\theta\|u_{n+1} - u_n\|^2 \\
&\quad + [\frac{\theta\mu_{n+1}}{\delta\mu_n} + 2\eta(\varepsilon+1)\frac{\mu_{n+1}}{\mu_{n+2}}]\|u_{n+1} - v_n\|^2 + 2\mu_n\delta\langle Au^*, u_n - u^*\rangle \\
&\leq \|u_n - u^*\|^2 - \theta\|u_{n-1} - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2 \\
&\quad + [\frac{\theta\mu_n}{\delta\mu_{n-1}} + 2\eta(\varepsilon+1)\frac{\mu_n}{\mu_{n+1}}]\|u_n - v_{n-1}\|^2 + 2\mu_{n-1}\delta\langle Au^*, u_{n-1} - u^*\rangle \\
&\quad - [1 - 3\theta - \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1} - u_n\|^2 \\
&\quad - [\frac{\mu_n}{\delta\mu_{n-1}} - \frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} - 2\eta\varepsilon^2\frac{\mu_n}{\mu_{n+1}} \\
&\quad - \frac{\theta\mu_{n+1}}{\delta\mu_n} - 2\eta(\varepsilon+1)\frac{\mu_{n+1}}{\mu_{n+2}}]\|u_{n+1} - v_n\|^2 \\
&\quad - [\frac{\mu_n}{\delta\mu_{n-1}} - \frac{\theta\mu_n}{\delta^2\mu_{n-1}} - 2\eta\frac{\varepsilon+1}{\varepsilon}\frac{\mu_n}{\mu_{n+1}}]\|u_n - v_n\|^2.
\end{aligned} \tag{33}$$

Let

$$\begin{aligned}
\beta_n = \max\{&\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}} + 2\eta\varepsilon^2\frac{\mu_n}{\mu_{n+1}} + \frac{\theta\mu_{n+1}}{\delta\mu_n} + 2\eta(\varepsilon+1)\frac{\mu_{n+1}}{\mu_{n+2}}, \\
&\frac{\theta\mu_n}{\delta^2\mu_{n-1}} + 2\eta\frac{\varepsilon+1}{\varepsilon}\frac{\mu_n}{\mu_{n+1}}\}.
\end{aligned} \tag{34}$$

Observe that

$$\overline{\eta} \leq \min\{\frac{\delta - (1+2\delta)\theta}{2(\varepsilon^2+\varepsilon+1)\delta^2}, \frac{(\delta-\theta)}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}\}.$$

We know that

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{\mu_n}{\mu_{n-1}\delta}-\beta_n\right)\\
&=\lim_{n\to\infty}\min\{\frac{\mu_n}{\mu_{n-1}\delta}-[\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}+2\eta\varepsilon^2\frac{\mu_n}{\mu_{n+1}}+\frac{\theta\mu_{n+1}}{\delta\mu_n}+2\eta(\varepsilon+1)\frac{\mu_{n+1}}{\mu_{n+2}}],\\
&\frac{\mu_n}{\mu_{n-1}\delta}-[\frac{\theta\mu_n}{\delta^2\mu_{n-1}}+2\eta\frac{\varepsilon+1}{\varepsilon}\frac{\mu_n}{\mu_{n+1}}]\}.\\
&=\min\{\frac{1}{\delta}-[\frac{(\delta+1)\theta}{\delta^2}+2\eta\varepsilon^2+\frac{\theta}{\delta}+2\eta(\varepsilon+1)],\frac{1}{\delta}-[\frac{\theta}{\delta^2}+2\eta\frac{\varepsilon+1}{\varepsilon}]\}\\
&>\min\{\frac{1}{\delta}-[\frac{(\delta+1)\theta}{\delta^2}+2\overline{\eta}\varepsilon^2+\frac{\theta}{\delta}+2\overline{\eta}(\varepsilon+1)],\frac{1}{\delta}-[\frac{\theta}{\delta^2}+2\overline{\eta}\frac{\varepsilon+1}{\varepsilon}]\}\\
&\geq 0.
\end{aligned}
\tag{35}
$$

So, there exists $n_1\in\mathbb{N}$ such that

$$\frac{\mu_n}{\mu_{n-1}\delta}-\beta_n>0,\quad \forall n\geq n_1.$$

By the facts that

$$
\begin{aligned}
2(\|u_n-v_n\|^2+\|u_{n+1}-v_n\|^2)&=\|u_n+u_{n+1}-2v_n\|^2+\|u_n-u_{n+1}\|^2\\
&\geq\|u_n-u_{n+1}\|^2,
\end{aligned}
$$

$\forall n\geq n_1$, we obtain from (33) that

$$
\begin{aligned}
s_{n+1}\leq s_n&-[1-3\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1}-u_n\|^2\\
&-[\frac{\mu_n}{\mu_{n-1}\delta}-\beta_n](\|u_n-v_n\|^2+\|u_{n+1}-v_n\|^2)\\
&\leq s_n-[1-3\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}+\frac{\mu_n}{2\mu_{n-1}\delta}-\frac{\beta_n}{2}]\|u_{n+1}-u_n\|^2.
\end{aligned}
\tag{36}
$$

Owing to $\lim_{n\to\infty}\mu_n=\mu$, we get

$$
\begin{aligned}
\lim_{n\to\infty}\beta_n&=\lim_{n\to\infty}\max\{\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}+2\eta\varepsilon^2\frac{\mu_n}{\mu_{n+1}}+\frac{\theta\mu_{n+1}}{\delta\mu_n}+2\eta(\varepsilon+1)\frac{\mu_{n+1}}{\mu_{n+2}},\\
&\quad\frac{\theta\mu_n}{\delta^2\mu_{n-1}}+2\eta\frac{\varepsilon+1}{\varepsilon}\frac{\mu_n}{\mu_{n+1}}\}\\
&=\max\{\frac{(\delta+1)\theta}{\delta^2}+2\eta\varepsilon^2+\frac{\theta}{\delta}+2\eta(\varepsilon+1),\frac{\theta}{\delta^2}+2\eta\frac{\varepsilon+1}{\varepsilon}\}\\
&=\max\{\frac{(2\delta+1)\theta}{\delta^2}+2\eta(\varepsilon^2+\varepsilon+1),\frac{\theta}{\delta^2}+2\eta\frac{\varepsilon+1}{\varepsilon}\}.
\end{aligned}
$$

Observe that

$$\overline{\eta} \leq \min\{\frac{2\delta^2(1-3\theta)-\delta+\theta-2\delta\theta}{2\delta^2(\varepsilon^2+\varepsilon+1)}, \frac{2\delta^2(1-3\theta)-\delta+\theta}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}\}.$$

Then

$$\begin{aligned}
&\lim_{n\to\infty}[1-3\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}+\frac{\mu_n}{2\mu_{n-1}\delta}-\frac{\beta_n}{2}]\\
&=\lim_{n\to\infty}[1-3\theta-\frac{(\delta-\theta)}{\delta^2}+\frac{1}{2\delta}-\frac{\beta_n}{2}]\\
&=\min\{1-3\theta-\frac{(\delta-\theta)}{\delta^2}+\frac{1}{2\delta}-\frac{(2\delta+1)\theta}{2\delta^2}-\eta(\varepsilon^2+\varepsilon+1),\\
&\quad 1-3\theta-\frac{(\delta-\theta)}{\delta^2}+\frac{1}{2\delta}-\frac{\theta}{2\delta^2}-\eta\frac{\varepsilon+1}{\varepsilon}\}\\
&>\min\{1-3\theta-\frac{(\delta-\theta)}{\delta^2}+\frac{1}{2\delta}-\frac{(2\delta+1)\theta}{2\delta^2}-\overline{\eta}(\varepsilon^2+\varepsilon+1),\\
&\quad 1-3\theta-\frac{(\delta-\theta)}{\delta^2}+\frac{1}{2\delta}-\frac{\theta}{2\delta^2}-\overline{\eta}\frac{\varepsilon+1}{\varepsilon}\}\\
&\geq 0.
\end{aligned}$$

Therefore there exists $n_2 \in \mathbb{N}$ such that

$$t_n = [1-3\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}+\frac{\mu_n}{2\mu_{n-1}\delta}-\frac{\beta_n}{2}]\|u_{n+1}-u_n\|^2 \geq 0, \quad \forall n \geq n_2.$$

So let $n_0 = \max\{n_1, n_2\}$, we have

$$s_{n+1} \leq s_n - t_n, \quad \forall n \geq n_0.$$

This completes the proof.

Theorem 1 *Suppose that the conditions* $(C1)$*-*$(C5)$ *are satisfied and* $\{u_n\}$ *is a sequence generated by Algorithm 1. Then* $\{u_n\}$ *weakly converges to a point in* S*.*

Proof From Lemma 5, we know that

$$\begin{aligned}
s_n &= \|u_n-u^*\|^2-\theta\|u_{n-1}-u^*\|^2+2\theta\|u_n-u_{n-1}\|^2\\
&\quad +[\frac{\theta\mu_n}{\delta\mu_{n-1}}+2\eta(\varepsilon+1)\frac{\mu_n}{\mu_{n+1}}]\|u_n-v_{n-1}\|^2+2\mu_{n-1}\delta\langle Au^*, u_{n-1}-u^*\rangle.
\end{aligned}$$

First, we need to prove that $s_n \geq 0$. Since

$$[\frac{\theta\mu_n}{\delta\mu_{n-1}}+2\eta(\varepsilon+1)\frac{\mu_n}{\mu_{n+1}}]\|u_n-v_{n-1}\|^2+2\mu_{n-1}\delta\langle Au^*, u_{n-1}-u^*\rangle \geq 0,$$

for any $n \geq n_0$, we get that

$$s_n \geq \|u_n - u^*\|^2 - \theta\|u_{n-1} - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2. \tag{37}$$

By Lemma 1, we have

$$\begin{aligned}\|u_{n-1} - u^*\|^2 &= \|(u_{n-1} - u_n) + (u_n - u^*)\|^2 \\ &= \|u_{n-1} - u_n\|^2 + \|u_n - u^*\|^2 + 2\langle u_{n-1} - u_n, u_n - u^*\rangle \\ &\leq 2\|u_{n-1} - u_n\|^2 + 2\|u_n - u^*\|^2.\end{aligned} \tag{38}$$

So from (37), we obtain

$$\begin{aligned}s_n &\geq \|u_n - u^*\|^2 - 2\theta\|u_{n-1} - u_n\|^2 - 2\theta\|u_n - u^*\|^2 + 2\theta\|u_n - u_{n-1}\|^2 \\ &= (1 - 2\theta)\|u_n - u^*\|^2, \quad \forall n \geq n_0.\end{aligned} \tag{39}$$

Hence for any $n \geq n_0$, we have $s_n \geq 0$. This means that $\lim_{n\to\infty} s_n$ exists and $\lim_{n\to\infty} t_n = 0$. From Lemma 5, we get

$$\lim_{n\to\infty} \|u_{n+1} - u_n\| = 0.$$

By (39), we know that the sequence $\{u_n\}$ is bounded. At the same time, from $v_n = u_n + \delta(u_n - u_{n-1})$ and $\omega_n = u_n + \theta(u_n - u_{n-1})$, we have that

$$\lim_{n\to\infty} \|v_n - u_n\| = \lim_{n\to\infty} \|\omega_n - u_n\| = 0. \tag{40}$$

Therefore

$$\|u_{n+1} - v_n\| \leq \|u_{n+1} - u_n\| + \|v_n - u_n\| \to 0, \quad n \to \infty. \tag{41}$$

We obtain that

$$\lim_{n\to\infty} \|u_n - u^*\|^2 - \theta\|u_{n-1} - u^*\|^2 + 2\mu_{n-1}\delta\langle Au^*, u_{n-1} - u^*\rangle \tag{42}$$

exists. Noticing the fact that the sequence $\{u_n\}$ is bounded, we have $u_{n_k} \subset u_n$ and $u_{n_k} \rightharpoonup z \in H$. If we have another $v_{n_k} \subset u_n$, then $v_{n_k} \rightharpoonup z$. Next, we prove that $z \in S$. By Definition 1 and monotonicity of A, we have that

$$\begin{aligned}0 &\leq \langle u_{n_k+1} - \omega_{n_k}, v - u_{n_k+1}\rangle + \mu_{n_k}\langle Av_{n_k}, v - u_{n_k+1}\rangle \\ &= \langle u_{n_k+1} - \omega_{n_k}, v - u_{n_k+1}\rangle + \mu_{n_k}\langle Av_{n_k}, v - v_{n_k}\rangle + \mu_{n_k}\langle Av_{n_k}, v_{n_k} - u_{n_k+1}\rangle \\ &\leq \langle u_{n_k+1} - \omega_{n_k}, v - u_{n_k+1}\rangle + \mu_{n_k}\langle Av, v - v_{n_k}\rangle + \mu_{n_k}\langle Av_{n_k}, v_{n_k} - u_{n_k+1}\rangle, \\ &\quad \forall v \in C.\end{aligned} \tag{43}$$

Let $k \to \infty$ in (43), we get

$$\langle Av, v - z\rangle \geq 0, \quad \forall v \in C. \tag{44}$$

From Lemma 2, $z \in S$. Assume that z_1 is a weak cluster point of $\{u_n\}$ and z_2 is another weak cluster point of $\{u_n\}$, $z_1 \neq z_2$. $\{u_{n_k}\}$ and $\{u_{n_l}\}$ are two subsequences of $\{u_n\}$ such that $u_{n_k} \rightharpoonup z_1$ and $u_{n_l} \rightharpoonup z_2$. We need to prove that $u_n \rightharpoonup z$. For $z_1, z_2 \in S$, we get

$$\begin{aligned}
&2\langle u_n, z_2 - z_1\rangle \\
&= \|u_n\|^2 + \|z_2 - z_1\|^2 - \|u_n - z_2 + z_1\|^2 \\
&= \|u_n\|^2 + \|z_2 - z_1\|^2 - \|u_n - z_2\|^2 - \|z_1\|^2 - 2\langle u_n - z_2, z_1\rangle \\
&= \|u_n - z_1 + z_1\|^2 + \|z_2 - z_1\|^2 - \|u_n - z_2\|^2 - \|z_1\|^2 - 2\langle u_n - z_2, z_1\rangle \\
&= \|u_n - z_1\|^2 + 2\langle u_n - z_1, z_1\rangle + \|z_2 - z_1\|^2 - \|u_n - z_2\|^2 - 2\langle u_n - z_2, z_1\rangle \\
&= \|u_n - z_1\|^2 - \|u_n - z_2\|^2 + 2\langle z_2 - z_1, z_1\rangle + \|z_2 - z_1\|^2 \\
&= \|u_n - z_1\|^2 - \|u_n - z_2\|^2 - \|z_1\|^2 + \|z_2\|^2.
\end{aligned} \tag{45}$$

Let $u_{n-1} = u_n$ and put it into (45), we obtain

$$2\langle u_{n-1}, z_2 - z_1\rangle = \|u_{n-1} - z_1\|^2 - \|u_{n-1} - z_2\|^2 - \|z_1\|^2 + \|z_2\|^2. \tag{46}$$

Hence

$$2\langle -\theta u_{n-1}, z_2 - z_1\rangle = -\theta\|u_{n-1} - z_1\|^2 + \theta\|u_{n-1} - z_2\|^2 + \theta\|z_1\|^2 - \theta\|z_2\|^2. \tag{47}$$

Combining (45) and (47), we gain

$$\begin{aligned}
&2\langle u_n - \theta u_{n-1}, z_2 - z_1\rangle \\
&= (\|u_n - z_1\|^2 - \theta\|u_{n-1} - z_1\|^2) \\
&\quad - (\|u_n - z_2\|^2 - \theta\|u_{n-1} - z_2\|^2) + (1-\theta)(\|z_2\|^2 - \|z_1\|^2).
\end{aligned} \tag{48}$$

By (42), we get that

$$\lim_{n\to\infty} \|u_n - z_1\|^2 - \theta\|u_{n-1} - z_1\|^2 + 2\mu_{n-1}\delta\langle Az_1, u_{n-1} - z_1\rangle \tag{49}$$

and

$$\lim_{n\to\infty} \|u_n - z_2\|^2 - \theta\|u_{n-1} - z_2\|^2 + 2\mu_{n-1}\delta\langle Az_2, u_{n-1} - z_2\rangle \tag{50}$$

exist. Therefore

$$\lim_{n\to\infty}[\langle u_n - \theta u_{n-1}, z_2 - z_1\rangle + \mu_{n-1}\delta(\langle Az_1, u_{n-1} - z_1\rangle - \langle Az_2, u_{n-1} - z_2\rangle)] \tag{51}$$

exists. Owing to $\lim_{n\to\infty}\mu_n = \mu$, we also get that

$$\begin{aligned}
&\langle z_1 - \theta z_1, z_2 - z_1\rangle + \delta\mu\langle Az_2, z_2 - z_1\rangle \\
&= \lim_{k\to\infty}[\langle u_{n_k} - \theta u_{n_k-1}, z_2 - z_1\rangle \\
&\quad + \mu_{n_k-1}\delta(\langle Az_1, u_{n_k-1} - z_1\rangle - \langle Az_2, u_{n_k-1} - z_2\rangle)] \\
&= \lim_{n\to\infty}[\langle u_n - \theta u_{n-1}, z_2 - z_1\rangle \\
&\quad + \mu_{n-1}\delta(\langle Az_1, u_{n-1} - z_1\rangle - \langle Az_2, u_{n-1} - z_2\rangle)] \\
&= \lim_{l\to\infty}[\langle u_{n_l} - \theta u_{n_l-1}, z_2 - z_1\rangle \\
&\quad + \mu_{n_l-1}\delta(\langle Az_1, u_{n_l-1} - z_1\rangle - \langle Az_2, u_{n_l-1} - z_2\rangle)] \\
&= \langle z_2 - \theta z_2, z_2 - z_1\rangle + \delta\mu\langle Az_1, z_2 - z_1\rangle.
\end{aligned} \tag{52}$$

This means that

$$(1-\theta)\|z_2 - z_1\|^2 + \delta\mu\langle Az_1 - Az_2, z_2 - z_1\rangle = 0. \tag{53}$$

We know that $z_1, z_2 \in S$, so

$$\langle Az_1 - Az_2, z_2 - z_1\rangle = \langle Az_1, z_2 - z_1\rangle + \langle Az_2, z_1 - z_2\rangle \geq 0. \tag{54}$$

By the monotonicity of A, we have

$$\langle Az_1 - Az_2, z_2 - z_1\rangle \leq 0. \tag{55}$$

Hence

$$\langle Az_1 - Az_2, z_2 - z_1\rangle = 0. \tag{56}$$

Also $\theta \in (0, 1)$, from (53) and (56), we attain $z_2 = z_1$. The proof is finshed.

4.2 Linear convergence

Suppose that the mapping A is γ-strongly monotone, that is, if for some $\gamma > 0$, we have

$$\langle Au - Av, u - v\rangle \geq \gamma\|u - v\|^2, \quad \forall u, v \in H. \tag{57}$$

Under this assumption, we carry out linear convergence analysis of the Algorithm 1.

Theorem 2 *Suppose that the conditions* $(C1)$-$(C3)$ *are satisfied and Lipschitz constant* $L > 0$. *Choose* $\theta \in [0, \infty)$, $\eta \in (0, \overline{\eta})$, $\delta > 1$ *and* $\varepsilon > 0$ *such that* $2\delta^2(1-3\theta) > \delta - \theta + 2\delta\theta$ *and* $0 \leq \theta < \min\{2\mu_0\gamma\delta, \frac{4\varepsilon\eta\gamma\delta}{L}, \frac{\delta}{2\delta+1}\}$, *where*

$$\overline{\eta} = \min\Big\{\frac{\delta-(1+2\delta)\theta}{2\delta^2(\varepsilon^2+\varepsilon+1)}, \frac{\delta-(\delta+1)\theta}{2\varepsilon^2\delta^2}, \frac{\delta-3\theta}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}, \frac{2\delta^2(1-3\theta)-\delta+\theta-2\delta\theta}{2\delta^2(\varepsilon^2+\varepsilon+1)}, \frac{2\delta^2(1-3\theta)-\delta+\theta}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}\Big\}.$$

Suppose that $\{u_n\}$ *is a sequence, which is generated by Algorithm 1. Then* $\{u_n\}$ *linearly converges to a point in* S.

Proof Let's choose u^* as a unique point of S. By Lemma 1, we have

$$\begin{aligned}\|v_n-u^*\|^2 &= \|(1+\delta)(u_n-u^*)-\delta(u_{n-1}-u^*)\|^2 \\ &= (1+\delta)\|u_n-u^*\|^2-\delta\|u_{n-1}-u^*\|^2+\delta(1+\delta)\|u_n-u_{n-1}\|^2 \\ &\geq (1+\delta)\|u_n-u^*\|^2-\delta\|u_{n-1}-u^*\|^2.\end{aligned} \tag{58}$$

Combining (57) and (58), we obtain

$$\begin{aligned}&2\mu_n[\langle Av_n-Au^*, v_n-u^*\rangle-\gamma((1+\delta)\|u_n-u^*\|^2 \\ &\quad -\delta\|u_{n-1}-u^*\|^2+\delta(1+\delta)\|u_n-u_{n-1}\|^2)] \\ &= 2\mu_n[\langle Av_n-Au^*, v_n-u^*\rangle-\gamma\|v_n-u^*\|^2] \\ &\geq 0.\end{aligned} \tag{59}$$

From (21), we know that

$$\begin{aligned}&\|u_{n+1}-u^*\|^2 \\ &\leq \|\omega_n-u^*\|^2-\|u_{n+1}-\omega_n\|^2+2\mu_n\langle Av_{n-1}-Av_n, u_{n+1}-v_n\rangle \\ &\quad -2\mu_n\langle Av_n, v_n-u^*\rangle+\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1}-u_n\|^2 \\ &\quad +(\frac{\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}})\|u_n-v_n\|^2+\frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n-v_{n-1}\|^2 \\ &\quad +[\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1}-v_n\|^2.\end{aligned} \tag{60}$$

Putting (59) into (60), we get

$$
\begin{aligned}
&\|u_{n+1}-u^*\|^2\\
&\le \|\omega_n-u^*\|^2-\|u_{n+1}-\omega_n\|^2-2\mu_n\gamma(1+\delta)\|u_n-u^*\|^2\\
&\quad+2\mu_n\gamma\delta\|u_{n-1}-u^*\|^2-2\mu_n\gamma\delta(1+\delta)\|u_n-u_{n-1}\|^2\\
&\quad+2\mu_n\langle Av_{n-1}-Av_n,u_{n+1}-v_n\rangle-2\mu_n\langle Au^*,v_n-u^*\rangle\\
&\quad+\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1}-u_n\|^2+(\frac{\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}})\|u_n-v_n\|^2\\
&\quad+\frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n-v_{n-1}\|^2+[\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1}-v_n\|^2.
\end{aligned}
\tag{61}
$$

Using (28), (29) in (61), we attain

$$
\begin{aligned}
&\|u_{n+1}-u^*\|^2\\
&\le (1+\theta-2\mu_n\gamma(1+\delta))\|u_n-u^*\|^2+(2\mu_n\gamma\delta-\theta)\|u_{n-1}-u^*\|^2\\
&\quad+2(\theta-\mu_n\gamma\delta(1+\delta))\|u_n-u_{n-1}\|^2-(1-\theta)\|u_{n+1}-u_n\|^2\\
&\quad+2\mu_n\langle Av_{n-1}-Av_n,u_{n+1}-v_n\rangle-2\mu_n\langle Au^*,v_n-u^*\rangle\\
&\quad+\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1}-u_n\|^2+(\frac{\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}})\|u_n-v_n\|^2\\
&\quad+\frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n-v_{n-1}\|^2+[\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1}-v_n\|^2.
\end{aligned}
\tag{62}
$$

Adding (22) and (25) to (62), we know

$$
\begin{aligned}
&\|u_{n+1}-u^*\|^2\\
&\le (1+\theta-2\mu_n\gamma(1+\delta))\|u_n-u^*\|^2+(2\mu_n\gamma\delta-\theta)\|u_{n-1}-u^*\|^2\\
&\quad+2(\theta-\mu_n\gamma\delta(1+\delta))\|u_n-u_{n-1}\|^2-(1-\theta)\|u_{n+1}-u_n\|^2\\
&\quad+\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1)\|u_n-v_{n-1}\|^2+\frac{\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon}\|u_n-v_n\|^2\\
&\quad+\frac{\mu_n}{\mu_{n+1}}2\eta\varepsilon^2\|u_{n+1}-v_n\|^2\\
&\quad-2\mu_n(1+\delta)\langle Au^*,u_n-u^*\rangle+2\mu_n\delta\langle Au^*,u_{n-1}-u^*\rangle\\
&\quad+\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}\|u_{n+1}-u_n\|^2+(\frac{\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}})\|u_n-v_n\|^2\\
&\quad+\frac{\theta\mu_n}{\delta\mu_{n-1}}\|u_n-v_{n-1}\|^2+[\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1}-v_n\|^2.
\end{aligned}
$$

Equivalently, we attain

$$
\begin{aligned}
&\|u_{n+1}-u^*\|^2\\
&\le (1+\theta-2\mu_n\gamma(1+\delta))\|u_n-u^*\|^2+(2\mu_n\gamma\delta-\theta)\|u_{n-1}-u^*\|^2\\
&\quad+2(\theta-\mu_n\gamma\delta(1+\delta))\|u_n-u_{n-1}\|^2+[\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}+\theta-1]\|u_{n+1}-u_n\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta\varepsilon^2+\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1}-v_n\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon}+\frac{\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_n-v_n\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1)+\frac{\theta\mu_n}{\delta\mu_{n-1}}]\|u_n-v_{n-1}\|^2\\
&\quad-2\mu_n(1+\delta)\langle Au^*,u_n-u^*\rangle+2\mu_n\delta\langle Au^*,u_{n-1}-u^*\rangle.
\end{aligned}
\tag{63}
$$

Observe that $v_n-u_n=\delta(u_n-u_{n-1})$, we have

$$
\|v_n-u_n\|^2=\delta^2\|u_n-u_{n-1}\|^2. \tag{64}
$$

Since $\mu_n\gamma\delta(1+\delta)>0$, we know

$$
\begin{aligned}
&\|u_{n+1}-u^*\|^2\\
&\le (1+\theta-2\mu_n\gamma(1+\delta))\|u_n-u^*\|^2+(2\mu_n\gamma\delta-\theta)\|u_{n-1}-u^*\|^2\\
&\quad+[\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}+\theta-1]\|u_{n+1}-u_n\|^2\\
&\quad+[2\theta+\delta^2(\frac{\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon}+\frac{\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}})]\|u_n-u_{n-1}\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta\varepsilon^2+\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}-\frac{\mu_n}{\delta\mu_{n-1}}]\|u_{n+1}-v_n\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1)+\frac{\theta\mu_n}{\delta\mu_{n-1}}]\|u_n-v_{n-1}\|^2\\
&\quad-2\mu_n(1+\delta)\langle Au^*,u_n-u^*\rangle+2\mu_n\delta\langle Au^*,u_{n-1}-u^*\rangle.
\end{aligned}
\tag{65}
$$

Hence from (65), we get

$$\begin{aligned}
&\|u_{n+1}-u^*\|^2\\
&\leq (1+\theta-2\mu_n\gamma(1+\delta))\|u_n-u^*\|^2+(2\mu_n\gamma\delta-\theta)\|u_{n-1}-u^*\|^2\\
&\quad-[1-\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1}-u_n\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1)+\frac{\theta\mu_n}{\delta\mu_{n-1}}]\|u_n-v_{n-1}\|^2\\
&\quad-[\frac{\delta\mu_n}{\mu_{n-1}}-2\theta-\frac{\theta\mu_n}{\mu_{n-1}}-\frac{\delta^2\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon}]\|u_n-u_{n-1}\|^2\\
&\quad-[\frac{\mu_n}{\delta\mu_{n-1}}-\frac{\mu_n}{\mu_{n+1}}2\eta\varepsilon^2-\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1}-v_n\|^2\\
&\quad-2\mu_n(1+\delta)\langle Au^*,u_n-u^*\rangle+2\mu_n\delta\langle Au^*,u_{n-1}-u^*\rangle.
\end{aligned}\tag{66}$$

Thus

$$\begin{aligned}
&\|u_{n+1}-u^*\|^2+2\mu_n(1+\delta)\langle Au^*,u_n-u^*\rangle\\
&\quad+[\frac{\mu_n}{\delta\mu_{n-1}}-\frac{\mu_n}{\mu_{n+1}}2\eta\varepsilon^2-\frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1}-v_n\|^2\\
&\leq (1+\theta-2\mu_n\gamma(1+\delta))\|u_n-u^*\|^2+(2\mu_n\gamma\delta-\theta)\|u_{n-1}-u^*\|^2\\
&\quad-[1-\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1}-u_n\|^2\\
&\quad+[\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1)+\frac{\theta\mu_n}{\delta\mu_{n-1}}]\|u_n-v_{n-1}\|^2\\
&\quad-[\frac{\delta\mu_n}{\mu_{n-1}}-2\theta-\frac{\theta\mu_n}{\mu_{n-1}}-\frac{\delta^2\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon}]\|u_n-u_{n-1}\|^2\\
&\quad+2\mu_n\delta\langle Au^*,u_{n-1}-u^*\rangle.
\end{aligned}\tag{67}$$

Owing to $\delta>1$ and $0\leq\theta<\frac{\delta}{2\delta+1}$, we get $\delta^2(1-\theta)-(\delta-\theta)>0$, then

$$\begin{aligned}
\lim_{n\to\infty}[1-\theta-\frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}}]&=1-\theta-\frac{(\delta-\theta)}{\delta^2}\\
&=\frac{\delta^2(1-\theta)-(\delta-\theta)}{\delta^2}\\
&>0.
\end{aligned}\tag{68}$$

Also, owing to $\eta < \overline{\eta} \leq \frac{\delta-3\theta}{2\delta^2}\frac{\varepsilon}{\varepsilon+1}$, we have

$$\begin{aligned}\lim_{n\to\infty}[\frac{\delta\mu_n}{\mu_{n-1}} - 2\theta - \frac{\theta\mu_n}{\mu_{n-1}} - \frac{\delta^2\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon}] &= \delta - 3\theta - \delta^2 2\eta\frac{\varepsilon+1}{\varepsilon} \\ &> \delta - 3\theta - \delta^2 2\overline{\eta}\frac{\varepsilon+1}{\varepsilon} \\ &\geq 0.\end{aligned} \tag{69}$$

So there exists $n_3 \in \mathbb{N}$ such that $\frac{\delta\mu_n}{\mu_{n-1}} - 2\theta - \frac{\theta\mu_n}{\mu_{n-1}} - \frac{\delta^2\mu_n}{\mu_{n+1}}2\eta\frac{\varepsilon+1}{\varepsilon} > 0$ and $1 - \theta - \frac{(\delta-\theta)\mu_n}{\delta^2\mu_{n-1}} > 0, \forall n \geq n_3$. It follows from (67) that

$$\begin{aligned}&\|u_{n+1} - u^*\|^2 + 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle \\ &\quad + [\frac{\mu_n}{\delta\mu_{n-1}} - \frac{\mu_n}{\mu_{n+1}}2\eta\varepsilon^2 - \frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1} - v_n\|^2 \\ &\leq (1 + \theta - 2\mu_n\gamma(1+\delta))\|u_n - u^*\|^2 + (2\mu_n\gamma\delta - \theta)\|u_{n-1} - u^*\|^2 \\ &\quad + [\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1) + \frac{\theta\mu_n}{\delta\mu_{n-1}}]\|u_n - v_{n-1}\|^2 \\ &\quad + 2\mu_n\delta\langle Au^*, u_{n-1} - u^*\rangle, \quad \forall n \geq n_3.\end{aligned} \tag{70}$$

Let

$$\xi_n = \max\{\frac{\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1) + \frac{\theta\mu_n}{\delta\mu_{n-1}}}{\frac{\mu_{n-1}}{\delta\mu_{n-2}} - \frac{\mu_{n-1}}{\mu_n}2\eta\varepsilon^2 - \frac{(\delta+1)\theta\mu_{n-1}}{\delta^2\mu_{n-2}}}, \frac{\mu_n\delta}{\mu_{n-1}(1+\delta)}\}. \tag{71}$$

We know that

$$\begin{aligned}&\lim_{n\to\infty}[\frac{\frac{\mu_n}{\mu_{n+1}}2\eta(\varepsilon+1) + \frac{\theta\mu_n}{\delta\mu_{n-1}}}{\frac{\mu_{n-1}}{\delta\mu_{n-2}} - \frac{\mu_{n-1}}{\mu_n}2\eta\varepsilon^2 - \frac{(\delta+1)\theta\mu_{n-1}}{\delta^2\mu_{n-2}}}] \\ &= \frac{2\eta(\varepsilon+1) + \frac{\theta}{\delta}}{\frac{1}{\delta} - 2\eta\varepsilon^2 - \frac{(\delta+1)\theta}{\delta^2}} \\ &= \frac{2\eta\delta^2(\varepsilon+1) + \theta\delta}{\delta - 2\eta\varepsilon^2\delta^2 - (\delta+1)\theta}.\end{aligned} \tag{72}$$

Owing to $\eta < \frac{\delta-(\delta+1)\theta}{2\varepsilon^2\delta^2}$ and $\eta < \frac{\delta-(1+2\delta)\theta}{2(\varepsilon^2+\varepsilon+1)\delta^2}$, we get

$$0 < \frac{2\eta\delta^2(\varepsilon+1) + \theta\delta}{\delta - 2\eta\varepsilon^2\delta^2 - (\delta+1)\theta} < 1. \tag{73}$$

Therefore there exists $n_4 \in \mathbb{N}$ such that

$$0 < \frac{\frac{\mu_n}{\mu_{n+1}} 2\eta(\varepsilon+1) + \frac{\theta\mu_n}{\delta\mu_{n-1}}}{\frac{\mu_{n-1}}{\delta\mu_{n-2}} - \frac{\mu_{n-1}}{\mu_n} 2\eta\varepsilon^2 - \frac{(\delta+1)\theta\mu_{n-1}}{\delta^2\mu_{n-2}}} < 1, \quad \forall n \geq n_4. \tag{74}$$

Let $\xi = \sup \xi_n$, it is obvious that $\xi \in (0, 1)$. Put $n_5 = \max\{n_3, n_4\}$, for any $n \geq n_5$, let

$$\begin{aligned} s_n &= \|u_n - u^*\|^2, \\ t_{n+1} &= [\frac{\mu_n}{\delta\mu_{n-1}} - \frac{\mu_n}{\mu_{n+1}} 2\eta\varepsilon^2 - \frac{(\delta+1)\theta\mu_n}{\delta^2\mu_{n-1}}]\|u_{n+1} - v_n\|^2 \\ &\quad + 2\mu_n(1+\delta)\langle Au^*, u_n - u^*\rangle. \end{aligned}$$

By (70), we attain

$$\begin{aligned} &s_{n+1} + t_{n+1} \\ &\leq (1 + \theta - 2\mu_n\gamma(1+\delta))s_n + (2\mu_n\gamma\delta - \theta)s_{n-1} + \xi_n t_n \\ &= [1 - (2\mu_n\gamma(1+\delta) - \theta)]s_n + \frac{(2\mu_n\gamma\delta - \theta)(2\mu_n\gamma(1+\delta) - \theta)}{2\mu_n\gamma(1+\delta) - \theta} s_{n-1} + \xi_n t_n. \end{aligned} \tag{75}$$

Let $\rho = \lim_{n\to\infty}(2\mu_n\gamma(1+\delta) - \theta)$. Owing to $0 \leq \theta < \min\{2\mu_0\gamma\delta, \frac{4\varepsilon\eta\gamma\delta}{L}\} \leq 2\mu_n\gamma\delta$, then $\rho > 0$. Therefore, there exists $n_6 \in \mathbb{N}$ and $\sigma \in (0, 1)$ such that

$$\frac{(2\mu_n\gamma\delta - \theta)(2\mu_n\gamma(1+\delta) - \theta)}{2\mu_n\gamma(1+\delta) - \theta} \leq \sigma\rho < \rho, \quad \forall n \geq n_6. \tag{76}$$

We know that $\mu_{n+1} \leq \mu_n$, let $n_7 = \max\{n_5, n_6\}$, then we obtain from (75) that

$$s_{n+1} + t_{n+1} \leq (1-\rho)s_n + \sigma\rho s_{n-1} + \xi t_n, \quad \forall n \geq n_7. \tag{77}$$

Let $r = \max\{\sigma, \xi\}$, then $r \in (0, 1)$. Thus we obtain

$$s_{n+1} + t_{n+1} \leq (1-\rho)s_n + r\rho s_{n-1} + r t_n, \quad \forall n \geq n_7.$$

Therefore, by Lemma 4, we have that $\{u_n\}$ linearly converges to a point in S. This completes the proof.

5 Modified algorithm

If appropriate parameters are selected, Algorithm 1 is able to effectively accelerate the convergence speed, but the main shortcoming of Algorithm 1 is that it can not completely restore some special algorithms in the literature. Therefore, we are going to do the following improvements and optimize our algorithm. The conditions $(C1)$-$(C3)$ of Algorithm 1 are still established, in this situation, we add two additional parameter conditions $(C6)$ and $(C7)$:

(C6) $\delta > 0, \varepsilon > 1$.

(C7) $\theta \in [0, \infty), 0 \leq \theta < \frac{\delta}{2\delta+1}$ and $2\delta^2(1-3\theta) > \delta - \theta + 2\delta\theta$.

Then, we provide a new modified algorithm as below:

Algorithm 2 Modified projected reflected gradient method with adaptive step size.

Initialization: Let $\mu_0 > 0$ and set $n := 0$. Assume

$$\overline{\eta} = \min\{\frac{\delta-(1+2\delta)\theta}{2\delta^2(\varepsilon^2+\varepsilon)}, \frac{(\delta-\theta)}{2\delta^2}\frac{\varepsilon-1}{\varepsilon},$$
$$\frac{2\delta^2(1-3\theta)-\delta+\theta-2\delta\theta}{2\delta^2(\varepsilon^2+\varepsilon)}, \frac{2\delta^2(1-3\theta)-\delta+\theta}{2\delta^2}\frac{\varepsilon-1}{\varepsilon}\},$$

choose $\eta \in (0, \overline{\eta})$ and Let $u_0 \in C,\ v_0, \omega_0 \in H$ be any three starting points.

Iterative Steps: u_{n+1} is determined via the previous v_n and ω_n as follows:

Step 1. Compute

$$\begin{cases} u_{n+1} = P_C(\omega_n - \mu_n A v_n), \\ v_{n+1} = u_{n+1} + \delta(u_{n+1} - u_n), \\ \omega_{n+1} = u_{n+1} + \theta(u_{n+1} - u_n). \end{cases} \tag{78}$$

If we have $\omega_n = v_n = u_{n+1}$, then Stop. Conversely, go to Step 2.

Step 2. Let

$$\rho_n := \varepsilon\|v_{n-1} - u_n\|^2 + \frac{\varepsilon}{\varepsilon-1}\|u_n - v_n\|^2 + \varepsilon^2\|u_{n+1} - v_n\|^2,$$

and turn the original step size into a new

$$\mu_{n+1} = \begin{cases} \mu_n, & \text{if}\langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle \leq 0, \\ \min\{\frac{\eta\rho_n}{\langle Av_n - Av_{n-1}, v_n - u_{n+1}\rangle}, \mu_n\}, & \text{otherwise.} \end{cases}$$

Step 3. Next: set $n := n + 1$ and go to **Step 1.**

Remark 2 Using the ideas of proof, which are similar to Algorithm 1, we can attain the weak convergence and linear convergence of Algorithm 2 without striking a blow.

Remark 3 Algorithm 2 is simplified to Algorithm 1 in [18], when $\theta = 0$ and $\varepsilon = \sqrt{2}$. Also, Algorithm 2 can be restored to Algorithm 1 in [11], only when $\varepsilon = \sqrt{2}$. If we choose $\theta = 0$, $\delta = 1$ and $\varepsilon = \sqrt{2}$, we can get Algorithm 3.1 in [7]. And then Algorithm 2 can be an extension of the other two algorithms in [8] and [9].

Remark 4 It is evident to discover that we can make numerous accelerated improvements and repeatedly optimize our algorithm by analogizing the step size rules of Algorithms 1 and 2. Meanwhile, those improvements are of great significance.

6 Numerical experiments

In this section, we present two numerical tests occurring in finite- and infinite-dimensional real Hilbert spaces to demonstrate the computational efficiency of the proposed algorithm and compare it with some recent algorithms in the literature [9, 11, 18]. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250S CPU @1.60GHz computer with RAM 8.00 GB.

Example 1 Consider the linear operator $A : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 20)$ in the form

$$A(u) = Mu + q,$$

where $q \in \mathbb{R}^m$ and $M = NN^{\mathsf{T}} + Q + D$, N is a $m \times m$ matrix, Q is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive symmetric definite). The feasible set C is given by

$$C = \left\{u \in \mathbb{R}^m : -2 \leq u_i \leq 5,\ i = 1, \dots, m\right\}.$$

It is clear that A is monotone and Lipschitz continuous with constant $L = \|M\|$. In this experiment, all entries of N, Q are generated randomly in $[-2, 2]$, D is generated randomly in $[0, 2]$ and $q = \mathbf{0}$. It is easy to check that the solution of the variational inequality problem is $u^* = \mathbf{0}$. Take $\varepsilon \in \{1, 2, 3, 4\}$, $\theta = 0.05$, $\delta = 1.1$, $\eta = 0.99\bar{\eta}$,

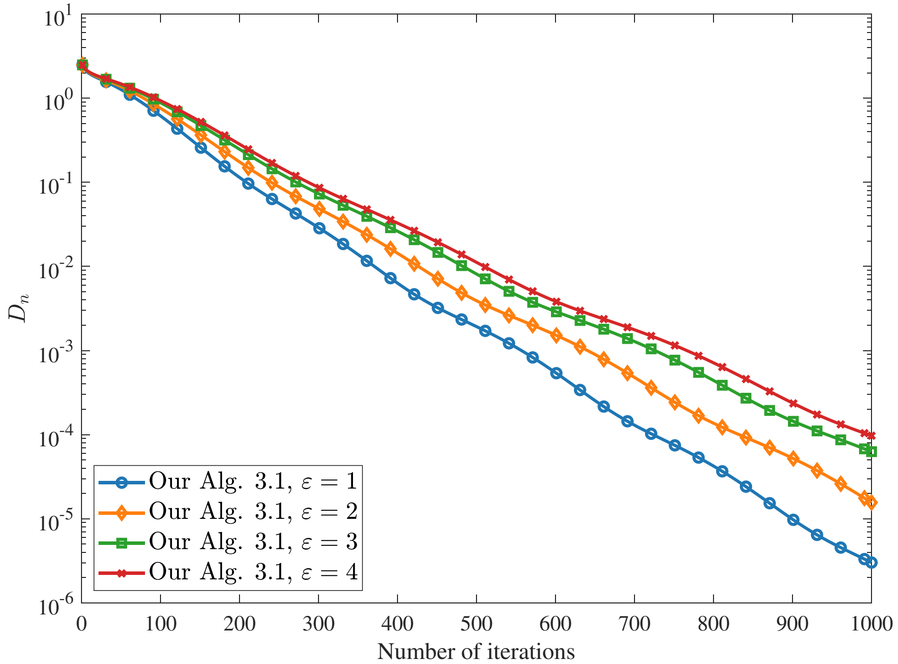

Fig. 1 Numerical results of our Algorithm 1 with different parameters ε for Example 1

and $\mu_0 = 0.01$ for our Algorithm 1(denoted as Alg.3.1). The maximum number of iterations 1000 is used as a stopping criterion. Figure 1 shows the numerical behavior of $D_n = \|u_n - u^*\|$ of our Algorithm 1 with different parameter ε.

Next, we compare the proposed Algorithm 1 with some related algorithms in the literature [9, 11, 18]. The parameters of these methods are set as follows.

- Our Algorithm 1: $\varepsilon = 1$, $\theta = 0.05$, $\delta = 1.1$, $\eta = 0.99\bar{\eta}$, and $\mu_0 = 0.01$.
- Iyiola and Shehu's Algorithm 1 (abbreviated as IS Alg. 1) [11]: $\theta = 0.05$, $\delta = 1.1$, $\mu = 0.99\bar{\mu}$, and $\lambda_0 = 0.01$.
- Thong, Gibali and Vuong's Algorithm 1 (abbreviated as TGV Alg. 1) [18]: $\theta = 0.6$, $\mu = 0.99\bar{\mu}$, and $\lambda_0 = 0.01$.
- Dong, He and Liu's Algorithm 1 (abbreviated as DHL Alg. 1) [9]: $\theta = 0.04$, $\delta = 1.1$, $\alpha = 0.99\bar{\alpha}$, and $\lambda_0 = 0.01$.

The numerical results of the proposed Algorithm 1 and the comparison algorithms are shown in Fig. 2.

Example 2 In this example, we consider our problem in the infinite-dimensional Hilbert space $H = L^2([0, 1])$ with inner product

$$\langle u, v\rangle := \int_0^1 u(t)v(t)\mathrm{d}t, \quad \forall u, v \in H,$$

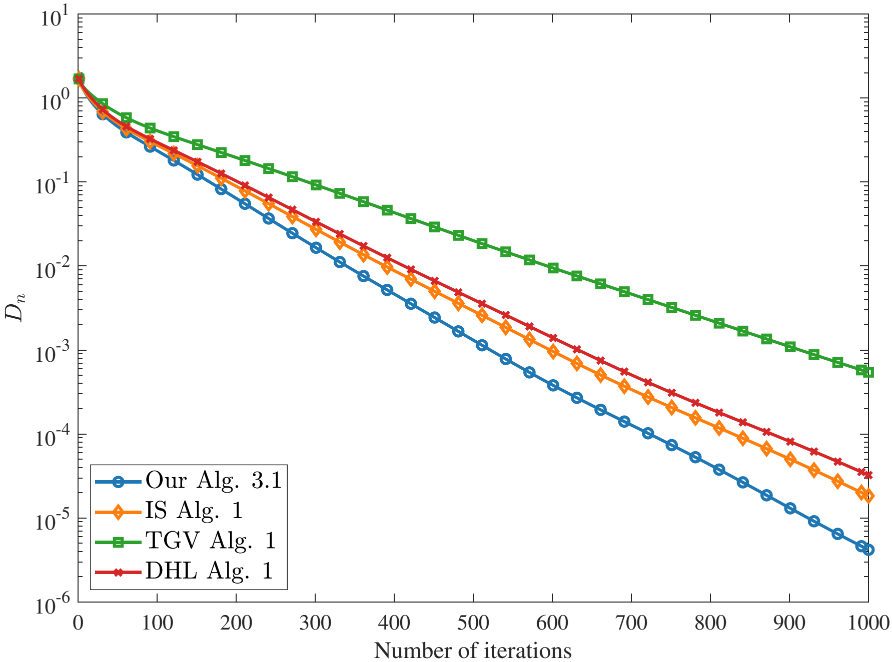

Fig. 2 Numerical results for all algorithms in Example 1

and norm

$$\|u\| := \left(\int_0^1 |u(t)|^2 \mathrm{d}t \right)^{1/2}, \quad \forall u \in H.$$

Let the feasible set be the unit ball $C := \{u \in H : \|u\| \leq 1\}$. Define an operator $A : C \to H$ by

$$(Au)(t) = \int_0^1 (u(t) - G(t,s)g(u(s)))\,\mathrm{d}s + h(t), \quad t \in [0,1],\ u \in C,$$

where

$$G(t,s) = \frac{2ts\mathrm{e}^{t+s}}{\mathrm{e}\sqrt{\mathrm{e}^2-1}}, \quad g(u) = \cos u, \quad h(t) = \frac{2t\mathrm{e}^t}{\mathrm{e}\sqrt{\mathrm{e}^2-1}}.$$

It is known that A is monotone and L-Lipschitz continuous with $L = 2$. The projection on C is inherently explicit, that is,

$$P_C(u) = \begin{cases} \frac{u}{\|u\|}, & \text{if } \|u\| > 1\,; \\ u, & \text{if } \|u\| \leq 1\,. \end{cases}$$

Through a straightforward calculation, we know that the solution of the variational inequality problem is $u^*(t) = 0$. Choose $\varepsilon \in \{1,2,3,4\}$, $\theta = 0.01$, $\delta = 1.1$, $\eta = 0.99\bar{\eta}$, $\mu_0 = 1$, and $u_0(t) = \omega_0(t) = v_0(t) = v_{-1}(t) = 10\sin(t)$ for our Algorithm 1.

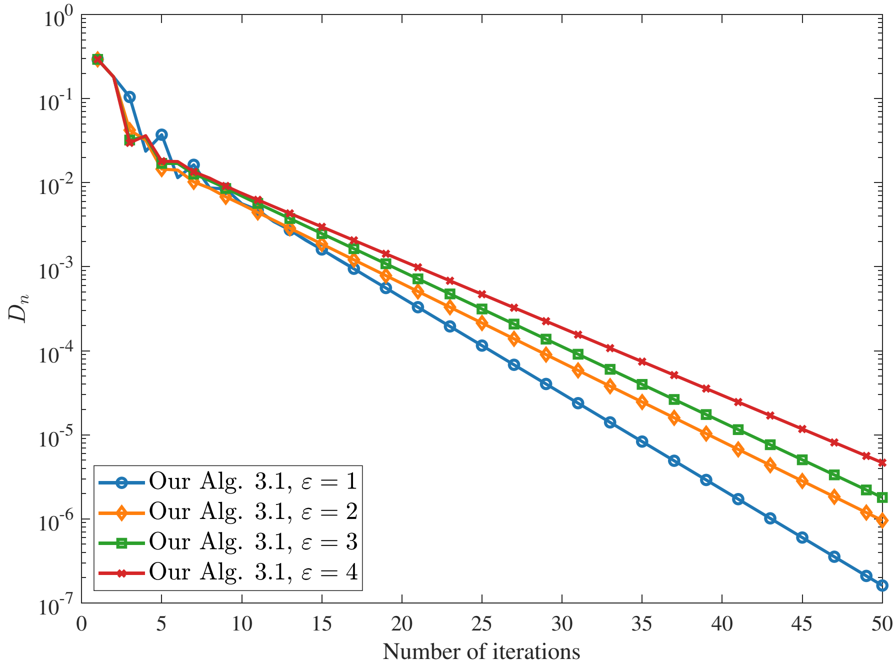

Fig. 3 Numerical results of our Algorithm 1 with different parameters ε for Example 2

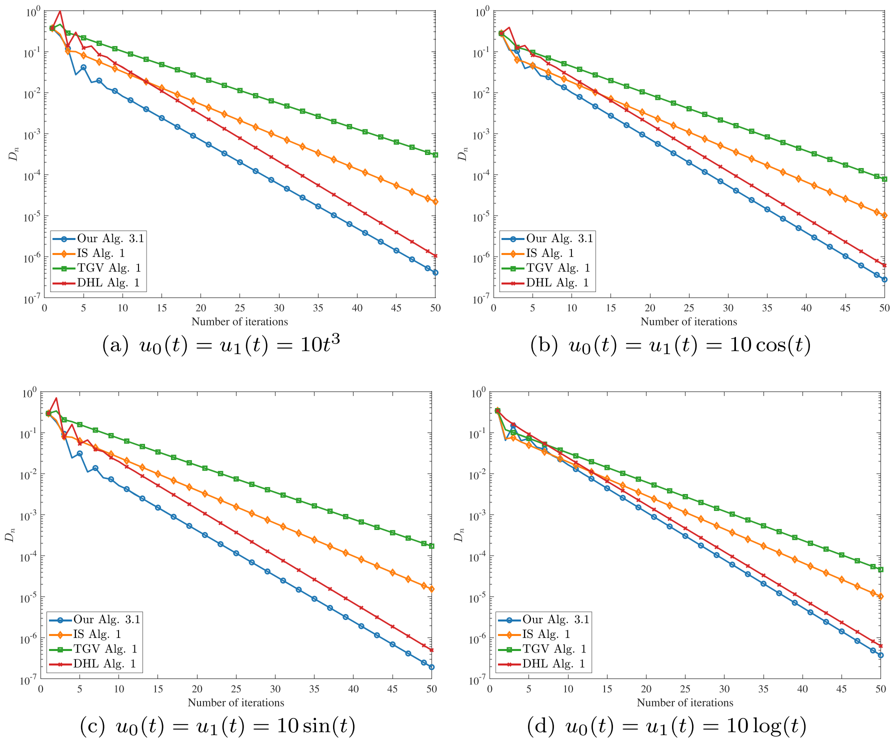

Fig. 4 Numerical results for all algorithms in Example 2

The maximum number of iterations 50 is used as a stopping criterion. Figure 3 shows the numerical behavior of $D_n = \|u_n(t) - u^*(t)\|$ of our Algorithm 1 with different parameter ε.

Next, we compare the proposed Algorithm 1 with some known algorithms in the literature [9, 11, 18]. The parameters of all algorithms are set as follows.

- Our Algorithm 1: $\varepsilon = 0.5$, $\theta = 0.01$, $\delta = 1.1$, $\eta = 0.99\bar{\eta}$, and $\mu_0 = 1$.
- Iyiola and Shehu's Algorithm 1 (abbreviated as IS Alg. 1) [11]: $\theta = 0.1$, $\delta = 1.1$, $\mu = 0.99\bar{\mu}$, and $\lambda_0 = 1$.
- Thong, Gibali and Vuong's Algorithm 1 (abbreviated as TGV Alg. 1) [18]: $\theta = 0.6$, $\mu = 0.99\bar{\mu}$, and $\lambda_0 = 1$.
- Dong, He and Liu's Algorithm 1 (abbreviated as DHL Alg. 1) [9]: $\theta = 0.04$, $\delta = 1.1$, $\alpha = 0.99\bar{\alpha}$, and $\lambda_0 = 1$.

The maximum number of iterations 50 is used as a common stopping criterion for all algorithms. With four different initial points, the numerical behavior of $D_n = \|u_n(t) - u^*(t)\|$ of all algorithms is described in Fig. 4.

We have the following observations for Examples 1 and 2.

1. From Figs. 1, 2, 3, and 4, it can be seen that the error D_n of the proposed Algorithm 1 converges to 0 as the number of iterations increases, which indicates that the

sequence of iterations generated by the proposed Algorithm 1 converges to the solution of the problem.
2. Our Algorithm 1 has a faster convergence speed when the appropriate parameters ε are chosen, as shown in Figs. 1 and 3. Therefore, a suitable parameter ε can be chosen to improve the convergence speed of the algorithm in practical applications.
3. It can be seen from Figs. 2 and 4 that our proposed Algorithm 1 has a faster convergence speed than some known algorithms in the literature [9, 11, 18] in the case of choosing some suitable parameters and that these results are not related to the choice of initial values.
4. Notice that our Algorithm 1 can work adaptively without knowing the prior information about the Lipschitz constants of the operators involved. If the Lipschitz constants of the involved operators are unknown, the fixed-step algorithms proposed in the literature [7, 20] for solving monotone variational inequality problems will not be available. Thus, the adaptive algorithm presented in this paper has a broader range of applications.

In conclusion, the algorithm proposed in this paper is efficient and robust.

7 Conclusions

In this work, we present a common version of projected reflected gradient method about solving variational inequalities in real Hilbert spaces. We certify that the sequence formed by our method converges weakly to a solution of the VIP. Linear convergence theorems are gained on the condition that the mapping A is strongly monotone. Also, we innovate a modified algorithm superior to our own algorithm under corresponding conditions. Numerical experiments that can show the efficiency and applicability of our methods are given in detail.

Acknowledgements The authors wish to thank the anonymous referees for their valuable comments and suggestions which lead to an improvement of this paper.

Author contribution Zhou and Cai wrote the main manuscript text, Tan and Dong finished the numerical examples. All authors reviewed the manuscript.

Funding This work was supported by the NSF of China (Grant No. 12171062) and the Natural Science Foundation of Chongqing (Grant No. CSTB2022NSCQ-JQX0004).

Availability of supporting data Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Ethics approval and consent to participate Not applicable

Consent for publication Not applicable

Human and animal ethics Not applicable

Competing interests The authors declare no competing interests.

References

1. Aubin, J.P., Ekeland, I.: Applied Nonlinear Analysis. Wiley, New York (1984)
2. Baiocchi, C., Capelo, A.: Variational and Quasivariational Inequalities. Applications to Free Boundary Problems. Wiley, New York (1984)
3. Facchinei, F., Pang, J.S.: Finite-Dimensional Variational Inequalities and Complementarity Problems, vol. I. Springer Series in Operations Research, Springer, New York (2003)
4. Kinderlehrer, D., Stampacchia, G.: An Introduction to Variational Inequalities and Their Applications. Academic Press, New York (1980)
5. Konnov, I.V.: Combined Relaxation Methods for Variational Inequalities. Springer, Berlin (2001)
6. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Ekonomikai Mat. Metody. **12**, 747–756 (1976)
7. Malitsky, Y.V.: Projected reflected gradient methods for monotone variational inequalities. SIAM J. Optim. **25**, 502–520 (2015)
8. Yang, J., Liu, H.: A modified projected gradient method for monotone variational inequalities. J. Optim. Theory Appl. **179**, 197–211 (2018)
9. Dong, Q.L., He, S., Liu, L.: A general inertial projected gradient method for variational inequality problems. Comput. Appl. Math. **40**, 168 (2021)
10. Shehu, Y., Iyiola, O.S., Reich, S.: A modified inertial subgradient extragradient method for solving variational inequalities. Optim. Eng. **23**(1), 421–449 (2022)
11. Iyiola, O.S., Shehu, Y.: Inertial version of generalized projected reflected gradient method. J. Sci. Comput. **93**(1), 24 (2022)
12. Yao, Y., Iyiola, O.S., Shehu, Y.: Subgradient extragradient method with double inertial steps for variational inequalities. J. Sci. Comput. **90**(2), 1–29 (2022)
13. Cai, G., Shehu, Y., Iyiola, O.S.: Inertial Tseng's extragradient method for solving variational inequality problems of pseudo-monotone and non-Lipschitz operators. Ind. Manag. Optim. **18**(4), 2873–2902 (2022)
14. Ceng, L.C., Yao, J.C.: Strong convergence theorem by an extragradient method for fixed point problems and variational inequality problems. Taiwan. J. Math. **10**, 1293–1303 (2006)
15. Thong, D.V., Hieu, D.V.: Inertial subgradient extragradient algorithms with line-search process for solving variational inequality problems and fixed point problems. Numer. Algorithms. **80**(4), 1283–1307 (2019)
16. Thong, D.V., Van Hieu, D., Rassias, T.M.: Self adaptive inertial subgradient extragradient algorithms for solving pseudomonotone variational inequality problems. Optim. Lett. **14**, 115–144 (2020)
17. Thong, D.V., Li, X.H., Dong, Q.L., Cho, Y.J., Rassias, T.M.: An inertial Popov's method for solving pseudomonotone variational inequalities. Optim. Lett. **15**, 757–777 (2021)
18. Thong, D.V., Gibali, A., Vuong, P.T.: An explicit algorithm for solving monotone variational inequalities. Appl. Numer. Math. **171**, 408–425 (2022)
19. Takahashi, W.: Nonlinear Functional Analysis. Yokohama Publishers, Yokohama (2000)
20. Censor, Y., Gibali, A., Reich, S.: The subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **148**, 318–335 (2011)
21. Popov, L.D.: A modification of the Arrow-Hurwicz method for finding saddle points. Math. Notes. **28**, 845–848 (1980)
22. Malitsky, Y.V., Semenov, V.V.: An extragradient algorithm for monotone variational inequalities. Cybernet. Systems Anal. **50**, 271–277 (2014)
23. Censor, Y., Gibali, A., Reich, S.: Extensions of Korpelevich's extragradient method for the variational inequality problem in Euclidean space. Optimization. **61**, 1119–1132 (2012)
24. Maingé, P.-E.: Numerical approach to monotone variational inequalities by a one-step projected reflected gradient method with line-search procedure. Comput. Math. Appl. **72**, 720–728 (2016)
25. Gibali, A., Reich, S., Zalas, R.: Outer approximation methods for solving variational inequalities in Hilbert space. Optimization. **66**, 417–437 (2017)
26. Maingé, P.-E., Gobinddass, M.L.: Convergence of one-step projected gradient methods for variational inequalities. J. Optim. Theory Appl. **171**, 146–168 (2016)
27. Malitsky, Y.: Golden ratio algorithms for variational inequalities. Math. Program. **184**, 383–410 (2020)

28. Ceng, L.C., Petrusel, A., Qin, X., Yao, J.C.: A modified inertial subgradient extragradient method for solving pseudomonotone variational inequalities and common fixed point problems. Fixed Point Theory. **21**, 93–108 (2020)
29. Ceng, L.C., Yuan, Q.: Composite inertial subgradient extragradient methods for variational inequalibties and fixed point problems. J. Inequal. Appl. **2019**(1), 1–20 (2019)
30. Censor, Y., Mansour, R.: Convergence analysis of processes with valiant projection operators in Hilbert space. J. Optim. Theory Appl. **176**, 35–56 (2018)
31. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2004)
32. Tan, B., Qin, X., Cho, S.Y.: Revisiting subgradient extragradient methods for solving variational inequalities. Numer. Algorithms **90**, 1593–1615 (2022)
33. Tan, B., Qin, X., Yao, J.C.: Strong convergence of inertial projection and contraction methods for pseudomonotone variational inequalities with applications to optimal control problems. J. Global Optim. **82**, 523–557 (2022)
34. Shehu, Y., Iyiola, O.S.: Weak convergence for variational inequalities with inertial-type method. Appl. Anal. **101**, 192–216 (2022)
35. Shehu, Y., Iyiola, O.S.: Projection methods with alternating inertial steps for variational inequalities: weak and linear convergence. Appl. Numer. Math. **157**, 315–337 (2020)