OPTIMIZATION
2024, VOL. 73, NO. 5, 1329–1354
https://doi.org/10.1080/02331934.2022.2157677

Inertial subgradient extragradient method for solving pseudomonotone equilibrium problems and fixed point problems in Hilbert spaces

Zhongbing Xie[a], Gang Cai[b] and Bing Tan [c,d]

[a]School of Mathematics and Statistics, Wuhan University, Wuhan, People's Republic of China; [b]School of Mathematical Sciences, Chongqing Normal University, Chongqing, People's Republic of China; [c]School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu, People's Republic of China; [d]Department of Mathematics, University of British Columbia, Kelowna, BC, Canada

ABSTRACT

This paper proposes a new inertial subgradient extragradient method for solving equilibrium problems with pseudomonotone and Lipschitz-type bifunctions and fixed point problems for nonexpansive mappings in real Hilbert spaces. Precisely, we prove that the sequence generated by proposed algorithm converges strongly to a common solution of equilibrium problems and fixed point problems. We use an effective self-adaptive step size rule to accelerate the convergence process of our proposed iterative algorithm. Moreover, some numerical results are given to show the effectiveness of the proposed algorithm. The results obtained in this paper extend and improve many recent ones in the literature.

ARTICLE HISTORY
Received 28 July 2021
Accepted 7 December 2022

KEYWORDS
Equilibrium problem; fixed point; pseudomonotone bifunction; strong convergence; subgradient extragradient method

MATHEMATICS SUBJECT CLASSIFICATION (2010)
47H05; 47H07; 47H10; 54H25

1. Introduction

We consider the following equilibrium problem (shortly, EP), also called Ky Fan's inequality due to his significant contribution [1] in 1972. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bifunction, then the equilibrium problem is stated as follows: find $x^* \in C$ such that

$$f(x^*, y) \geq 0, \quad \forall\, y \in C. \tag{1}$$

We denote by EP(f, C) the solution set of EP(1).

The EP has a wide range of applications in the field of mathematics. For example, it can be applied to solve variational inequality problems, fixed point problems, saddle point problems and Nash equilibrium problems (see, e.g. [2–17] and the references therein). At the same time, two momentous methods have

CONTACT Gang Cai caigang-aaaa@163.com

been proposed to solve the EP: proximal point method (shortly, PPM) [18,19] and auxiliary problem principle [20].

The PPM was originally proposed by Martinet [21] to solve variational inequality problem, and later Moudafi [18] applied it to solve monotone equilibrium problems. However, the PPM is limited by the inability to solve pseudomonotone equilibrium problem. In order to overcome this shortcoming, Flam et al. [22] and Tran et al. [23] successively introduced a proximal-like method which is also called the extragradient method (shortly, EGM). Precisely, the algorithm in Tran et al. [23] is as follows:

$$\begin{cases} u_0 \in C, \\ v_n = \arg\min_{y\in C} \left\{ \lambda f(u_n, y) + \frac{1}{2}\|y - u_n\|^2 \right\}, \\ u_{n+1} = \arg\min_{y\in C} \left\{ \lambda f(v_n, y) + \frac{1}{2}\|y - u_n\|^2 \right\}, \end{cases}$$

where $\lambda > 0$ is a suitable parameter, f is a pseudomonotone bifuction. They proved a weak convergence theorem of the iterative sequence $\{u_n\}$ generated by the above extragradient method. Since then, the EGM has attracted the attention of many authors, see, e.g. [24–34] and the references therein.

On the other hand, it is well known that the inertial technology can speed up the convergence rate of the related algorithms, so many authors apply it to various fields (see, e.g. [15,29,35–46] and the references therein). For instance, Thong and Hieu [42] applied inertial technology to solve the variational inequality problem in Hilbert space, Yao et al. [43] conducted a convergence analysis of the inertial iteration in the split feasibility problem, and Tan et al. [40] introduced inertial algorithm for solving split variational inclusion problem. For the equilibrium problem, Rehman et al. [29] proposed the following algorithm with inertia term for solving pseudomonotone equilibrium problem in real Hilbert space:

where f is a pseudomonotone operator satisfying the Lipschitz-type condition on Hilbert space $\mathcal{H}$. Rehman et al. [29] merged the inertial method and the EGM involving a new self-adaptive step size rule to obtain the weak convergence result of the generated sequence. Nevertheless, the self-adaptive rule in Algorithm 1 requires the Lipschitz constants L_1 and L_2 to be known in advance. We naturally think of improving this rule so that no prior knowledge of Lipschitz constants is needed. Furthermore, since weak convergence is not as good as strong convergence, we also consider a strong convergence theorem about equilibrium problems in Hilbert spaces.

In recent years, the fixed point problem (shortly, FPP) has also been a hot issue in mathematics research. The FPP is formulated as

$$\text{find } x \in \mathcal{H} \quad \text{such that } x \in F(T),$$

where $F(T) := \{x : x = Tx\}$ is the set of fixed points of T. At the same time, many researchers have proposed multifarious related methods for finding a common

Algorithm 1

Initialization: Choose $u_{-1}, u_0 \in \mathcal{H}$, $\rho \in (0,1)$, $0 < \sigma < \min\{\frac{1-3\theta}{(1-\theta)^2}, \frac{1}{2L_1}, \frac{1}{2L_2}\}$, $\mu \in (0,\sigma)$, $\lambda_0 > 0$, and a nondecreasing sequence $0 \le \theta_n \le \theta < \frac{1}{3}$.

Iterative Steps: Given u_{n-1}, u_n and λ_n are known for $n \ge 0$.

Step 1. Evaluate

$$v_n = \arg\min_{y \in C} \left\{ \lambda_n f(t_n, y) + \frac{1}{2}\|y - t_n\|^2 \right\},$$

where $t_n = u_n + \theta_n(u_n - u_{n-1})$. If $t_n = v_n$; STOP. Otherwise go to next step.

Step 2. Evaluate

$$u_{n+1} = \arg\min_{y \in C} \left\{ \mu\lambda_n f(v_n, y) + \frac{1}{2}\|y - t_n\|^2 \right\}.$$

Step 3. Next, the step size sequence λ_{n+1} is updated as follows:

$$\lambda_{n+1} = \min\left\{ \sigma, \frac{\mu f(v_n, u_{n+1})}{f(t_n, u_{n+1}) - f(t_n, v_n) - L_1\|t_n - v_n\|^2 - L_2\|u_{n+1} - v_n\|^2 + 1} \right\}.$$

Set $n := n + 1$ and return back to **Iterative steps**,

solution that belongs to the intersection of EP(f, C) and $F(T)$ (see, e.g. [47–52] and the references therein). For example, Yang et al. [50] proposed the following algorithm:

where f is a pseudomonotone operator satisfying the Lipschitz-type condition, T is a quasi-nonexpansive mapping and T_n is a half-space, which was first introduced by Censor et al. [53]. Under appropriate assumptions, Yang et al. [50] obtained that the sequence $\{u_n\}$ generated by Algorithm 2 converges strongly to a common solution of the EP and FPP. It is worth noting that the strong convergence theorem about Algorithm 2 does not need to know the Lipschitz constant, which is the main point of our consideration.

Motivated by the above works, in this paper, we prove a strong convergence theorem of inertial subgradient extragradient method for solving the EP and FPP in Hilbert spaces. Compared with Algorithm 2, we introduce a new parameter to improve the step size, which is meaningful through numerical examples. In order to obtain a strong convergence theorem, we add a contraction mapping to the iterative sequence, which is different from the sequence $\{u_{n+1}\}$ in Algorithm 2. In addition, the inertial technology is applied to accelerate the convergence speed of the proposed algorithm. Finally, several numerical experimental results show that our algorithm does have better convergence than other existing related algorithms.

Algorithm 2

Initialization: Take $\lambda_0 > 0$, $u_0 \in \mathcal{H}$, $\mu \in (0,1)$.
Iterative Steps: Given the current iterate u_n, calculate u_{n+1} as follows:
Step 1. Compute

$$v_n = \arg\min_{y \in C} \left\{ \lambda_n f(u_n, y) + \frac{1}{2}\|y - u_n\|^2 \right\}.$$

Step 2. Choose $z_n \in \partial_2 f(u_n, v_n)$ such that $u_n - \lambda_n z_n - v_n \in N_C(v_n)$, compute

$$w_n = \arg\min_{y \in T_n} \left\{ \lambda_n f(v_n, y) + \frac{1}{2}\|y - u_n\|^2 \right\},$$

where

$$T_n = \{x \in \mathcal{H} : \langle u_n - \lambda_n z_n - v_n, x - v_n \rangle \le 0\}.$$

Step 3. Compute $t_n = \alpha_n u_0 + (1 - \alpha_n) w_n$, $u_{n+1} = \beta_n w_n + (1 - \beta_n) T t_n$ and

$$\lambda_{n+1} = \begin{cases} \min \left\{ \dfrac{\mu}{2} \dfrac{\|u_n - v_n\|^2 + \|w_n - v_n\|^2}{f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)}, \lambda_n \right\}, & \text{if } f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n) > 0, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return to **Step 1**,

2. Preliminaries

We first recall some basic concepts and facts.

For any $x, y, z \in \mathcal{H}$, it is well known that

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y \rangle \tag{2}$$

and

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x - y\|^2 - \alpha\gamma\|x - z\|^2 - \beta\gamma\|y - z\|^2, \tag{3}$$

where $\alpha, \beta, \gamma \in [0,1]$ with $\alpha + \beta + \gamma = 1$.

Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. A bifunction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is said to be:

(i) monotone on C if

$$f(u, v) + f(v, u) \le 0, \quad \forall\, u, v \in C. \tag{4}$$

(ii) pseudomonotone on C if

$$f(u,v) \geq 0 \Longrightarrow f(v,u) \leq 0, \quad \forall\ u,v \in C. \tag{5}$$

(iii) satisfying a Lipschitz-type condition on C if there exist two positive constants c_1, c_2 such that

$$f(u,v) + f(v,w) \geq f(u,w) - c_1\|u-v\|^2 - c_2\|v-w\|^2, \quad \forall\, u,v,w \in C. \tag{6}$$

For every point $x \in H$, it is well known that there exists a unique nearest point in C, denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|$, $\forall\, y \in C$. P_C is called the metric projection of H onto C. In addition, the following inequality holds:

$$\langle P_C(u) - u, v - P_C(u)\rangle \geq 0, \quad \forall\, v \in C.$$

For any $u, v \in \mathcal{H}$, the subdifferential $\partial_2 f(u,v)$ of $f(u,\cdot)$ at v is defined by

$$\partial_2 f(u,v) = \{x \in \mathcal{H} : f(u,y) - f(u,v) \geq \langle x, y - v\rangle,\ \forall\, y \in \mathcal{H}\}. \tag{7}$$

Let $T : \mathcal{H} \to \mathcal{H}$ be a mapping with $F(T) \neq \emptyset$, where $F(T)$ is set of the fixed points of T. Then

(i) T is called nonexpansive if

$$\|Tu - Tv\| \leq \|u - v\|, \quad \forall\, u,v \in \mathcal{H}.$$

(ii) $I - T$ is called demiclosed at zero if $\{u_n\} \subset \mathcal{H}$, $u_n \rightharpoonup u$ and $\|Tu_n - u_n\| \to 0$ implies $u \in F(T)$.

In order to obtain the main results of this paper, we need the following lemmas.

Lemma 2.1 ([54]): *Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ and $h : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function on $\mathcal{H}$. Assume either that h is continuous at some point of C, or that there is an interior point of C where h is finite. Then, x^* is a solution to the following convex problem* $\min\{h(x) : x \in C\}$ *if and only if $0 \in \partial h(x^*) + N_C(x^*)$, where $\partial h(\cdot)$ denotes the subdifferential of h and $N_C(x^*)$ is the normal cone of C at x^*.*

Lemma 2.2 ([55]): *Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping and $\mathcal{H}$ be a real Hilbert space. Let $\{x_n\}$ be a sequence in $\mathcal{H}$ and x be a point in $\mathcal{H}$. Suppose that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then $x \in F(T)$.*

Lemma 2.3 ([56]): *Let $\{d_n\}$ be a sequence of non-negative real number such that there exists a subsequence $\{d_{n_j}\}$ of $\{d_n\}$ such that $d_{n_j} < d_{n_j+1}$ for all $j \in \mathbb{N}$. Then* *there exists a non-decreasing sequence* $\{m_k\} \subset \mathbb{N}$ *such that* $\lim_{k\to\infty} m_k = \infty$ *and the following properties are satisfied by all (sufficiently large) number* $k \in \mathbb{N}$*:*

$$d_{m_k} \leq d_{m_k+1} \quad \text{and} \quad d_k \leq d_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1, 2, \ldots, k\}$ *such that* $d_n < d_{n+1}$*.*

Lemma 2.4 ([57]): *Let* $\{d_n\}$ *be a sequence of nonnegative real numbers such that*

$$d_{n+1} \leq (1 - a_n)d_n + a_n b_n + c_n, \quad \forall\, n \geq 0,$$

where $\{a_n\}$*,* $\{b_n\}$ *and* $\{c_n\}$ *satisfy:*

(a) $\{a_n\} \subset [0, 1], \sum_{n=1}^{\infty} \alpha_n = \infty$;
(b) $\limsup_{n\to\infty} b_n \leq 0$;
(c) $c_n \geq 0\ (n \geq 0), \sum_{n=1}^{\infty} c_n < \infty$.

Then $\lim_{n\to\infty} d_n = 0$.

3. Main results

In this section, we propose a modified subgradient extragradient method for finding a common element of the solution sets of the equilibrium problem (EP) and the fixed point problem (FPP) in Hilbert spaces. The advantage of our method is that we use a new parameter to improve the step size in the proposed algorithm, and the proof of the convergence theorem does not require estimating the Lipschitz constants. In order to get a strong convergence result, we need the following assumptions.

Assume that the feasible set C is a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, the bifuction $f : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is pseudomonotone and satisfies the Lipschitz-type condition on $\mathcal{H}$, $f(u, \cdot)$ is subdifferentiable on $\mathcal{H}$ for any $u \in \mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping, $g : \mathcal{H} \to \mathcal{H}$ be a contraction mapping with a constant $\rho \in [0, 1)$ and the solution set $EP(f, C) \cap F(T) \neq \emptyset$. Suppose that the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\theta_n\}$ satisfy the following conditions:

(C1) $\{\alpha_n\} \subset (0, 1), \lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C2) $0 < \liminf_{n\to\infty}\{\beta_n\} \leq \limsup_{n\to\infty} \beta_n < 1$;
(C3) $\{\theta_n\} \subset [0, \theta)$ for some $\theta > 0$ such that $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\| = 0$ (see [44,45] for more details), where $t_n \in \mathcal{H}$ is a sequence in the following Algorithm.

Now, we introduce the following algorithm.

Remark 3.1: Based on Algorithms 1, 2 and other related results, our algorithm has some improvements in the following:

Algorithm 3

Initialization: Let $\tau_1 > 0$, $\mu \in (0,1)$, $k \in (0,1]$ and $t_0, t_1 \in \mathcal{H}$.
Iterative steps: Given the current iterates t_{n-1} and t_n $(n \geq 1)$.
Step 1. Evaluate

$$v_n = \arg\min_{y \in C} \left\{ \tau_n f(u_n, y) + \frac{1}{2}\|y - u_n\|^2 \right\},$$

where $u_n = t_n + \theta_n(t_n - t_{n-1})$. If $v_n = u_n$, then stop. Otherwise go to **Step 2**.
Step 2. Choose $z_n \in \partial_2 f(u_n, v_n)$ such that $u_n - \tau_n z_n - v_n \in N_C(v_n)$. Compute

$$w_n = \arg\min_{y \in T_n} \left\{ k\tau_n f(v_n, y) + \frac{1}{2}\|y - u_n\|^2 \right\},$$

where

$$T_n := \{x \in \mathcal{H} : \langle u_n - \tau_n z_n - v_n, x - v_n \rangle \leq 0\}.$$

Step 3. Calculate

$$t_{n+1} = \alpha_n g(t_n) + \beta_n t_n + (1 - \beta_n - \alpha_n) T w_n,$$

and

$$\tau_{n+1} = \begin{cases} \min \left\{ \dfrac{\mu}{2} \dfrac{\|u_n - v_n\|^2 + \|w_n - v_n\|^2}{f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)}, \tau_n \right\}, & \text{if } f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n) > 0, \\ \tau_n, & \text{otherwise.} \end{cases} \tag{8}$$

Set $n := n + 1$ and return back to **Step 1**.

(1) Following the self-adaptive rule in Algorithm 2, we can prove that the main theorem of this paper does not require knowledge of Lipschitz constants. On this basis, we improve the variable μ in Algorithm 1, which is analogous to k in our algorithm. If $k = 1$, it is the general situation; whereas if $k \in (0,1)$, the convergence process of our algorithm can be improved by the difference of the value of k, and it is reflected in the numerical example.
(2) In Algorithm 1, there is a relationship between the value of μ and the inertia term, which may limit the convergence effect. We separate this value from the inertia term, which will improve the convergence speed of Algorithm 3.
(3) We combine inertial subgradient extragradient method and viscosity iterative method to deal with fixed point problems and equilibrium problems, our

proposed iterative algorithm is new and different from Algorithm 2. Under some appropriate assumptions imposed on the parameters, we prove that the sequence generated by Algorithm 3 converges strongly to a common solution of the equilibrium problems and fixed point problems.

Lemma 3.1: *The sequence* $\{\tau_n\}$ *generated by* (8) *is non-increasing and*

$$\lim_{n\to\infty} \tau_n \geq \min\left\{\frac{\mu}{2\max\{c_1, c_2\}}, \tau_1\right\}.$$

Proof: From (8), it is clear to get the sequence $\{\tau_n\}$ is non-increasing. In addition, since f satisfies the Lipschitz-type condition on $\mathcal{H}$, we have

$$\begin{aligned}\frac{\mu}{2}\frac{\|u_n - v_n\|^2 + \|w_n - v_n\|^2}{f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)} &\geq \frac{\mu}{2}\frac{\|u_n - v_n\|^2 + \|w_n - v_n\|^2}{c_1\|u_n - v_n\|^2 + c_2\|v_n - w_n\|^2}\\ &\geq \frac{\mu}{2\max\{c_1, c_2\}}.\end{aligned}$$

Therefore, $\{\tau_n\}$ is a non-increasing sequence and lower bounded. Moreover, there exits $\lim_{n\to\infty} \tau_n = \tau \geq \min\{\frac{\mu}{2\max\{c_1,c_2\}}, \tau_1\}$. ∎

Lemma 3.2: *Let* $\{u_n\}, \{v_n\}$ *and* $\{w_n\}$ *be the sequences generated by Algorithm* 3. *Then*

$$\begin{aligned}\|w_n - p\|^2 &\leq \|u_n - p\|^2 - (1-k)\|u_n - w_n\|^2\\ &\quad - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n - v_n\|^2 - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2,\end{aligned}$$

for all $p \in EP(f, C)$.

Proof: From Lemma 2.1 and the definition of $\{w_n\}$, we have

$$0 \in \partial\left\{k\tau_n f(v_n, y) + \frac{1}{2}\|y - u_n\|^2\right\}(w_n) + N_{T_n}(w_n), \quad \forall\, y \in T_n.$$

It follows that there exist $s_n \in \partial_2 f(v_n, w_n)$ and $\overline{s_n} \in N_{T_n}(w_n)$ such that

$$k\tau_n s_n + w_n - u_n + \overline{s_n} = 0.$$

That is

$$\langle u_n - w_n, y - w_n\rangle = k\tau_n\langle s_n, y - w_n\rangle + \langle \overline{s_n}, y - w_n\rangle, \quad \forall\, y \in T_n.$$

Since $\overline{s_n} \in N_{T_n}(w_n)$, we obtain $\langle \overline{s_n}, y - w_n\rangle \leq 0$. Then

$$k\tau_n\langle s_n, y - w_n\rangle \geq \langle u_n - w_n, y - w_n\rangle, \quad \forall\, y \in T_n. \tag{9}$$

In addition, by the definition of subdifferential and $s_n \in \partial_2 f(v_n, w_n)$, we get

$$f(v_n, y) - f(v_n, w_n) \geq \langle s_n, y - w_n \rangle, \quad \forall\, y \in T_n. \tag{10}$$

Combining (9) and (10) we have

$$k\tau_n(f(v_n, y) - f(v_n, w_n)) \geq \langle u_n - w_n, y - w_n \rangle, \quad \forall\, y \in T_n. \tag{11}$$

Let $y := p \in EP(f, C) \subset C \subset T_n$, then

$$k\tau_n(f(v_n, p) - f(v_n, w_n)) \geq \langle u_n - w_n, p - w_n \rangle. \tag{12}$$

As $v_n \in C$ we have $f(p, v_n) \geq 0$. By the pseudomonotonicity of f we get $f(v_n, p) \leq 0$. Thus (12) can be transformed into

$$\langle u_n - w_n, w_n - p \rangle \geq k\tau_n f(v_n, w_n). \tag{13}$$

Similarly, since $z_n \in \partial_2 f(u_n, v_n)$, we obtain

$$f(u_n, z) - f(u_n, v_n) \geq \langle z_n, z - v_n \rangle, \quad \forall\, z \in \mathcal{H}.$$

Let $z := w_n$, then

$$f(u_n, w_n) - f(u_n, v_n) \geq \langle z_n, w_n - v_n \rangle. \tag{14}$$

By definition of T_n and $w_n \in T_n$, we have $\langle u_n - \tau_n z_n - v_n, w_n - v_n \rangle \leq 0$. This implies that

$$\tau_n \langle z_n, w_n - v_n \rangle \geq \langle u_n - v_n, w_n - v_n \rangle. \tag{15}$$

Combining (14) and (15), we get

$$\tau_n(f(u_n, w_n) - f(u_n, v_n)) \geq \langle u_n - v_n, w_n - v_n \rangle. \tag{16}$$

From (8), we obtain

$$\tau_{n+1}(f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)) \leq \frac{\mu}{2}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2),$$

or equivalently

$$\tau_n(f(u_n, w_n) - f(u_n, v_n) - f(v_n, w_n)) \leq \frac{\tau_n}{\tau_{n+1}} \frac{\mu}{2}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2). \tag{17}$$

Substituting (17) into (16), then

$$\langle u_n - v_n, w_n - v_n \rangle \leq \tau_n f(v_n, w_n) + \frac{\tau_n}{\tau_{n+1}} \frac{\mu}{2}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2). \tag{18}$$

Adding (13) and (18) we get

$$\langle u_n - v_n, w_n - v_n\rangle \le \frac{1}{k}\langle u_n - w_n, w_n - p\rangle + \frac{\tau_n}{\tau_{n+1}}\frac{\mu}{2}(\|u_n - v_n\|^2 + \|w_n - v_n\|^2). \tag{19}$$

On the other hand,

$$\begin{aligned} 2\langle u_n - v_n, w_n - v_n\rangle &= \|u_n - v_n\|^2 + \|w_n - v_n\|^2 - \|u_n - w_n\|^2, \\ 2\langle u_n - w_n, w_n - p\rangle &= \|u_n - p\|^2 - \|w_n - p\|^2 - \|u_n - w_n\|^2. \end{aligned} \tag{20}$$

Combining (19) and (20), we obtain

$$\|w_n - p\|^2 \le \|u_n - p\|^2 - (1-k)\|u_n - w_n\|^2 - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n - v_n\|^2 - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2.$$

The proof is completed. ■

Lemma 3.3: *The sequence $\{t_n\}$ generated by Algorithm 3 is bounded.*

Proof: From Lemma 3.1 and $k \in (0, 1]$, $\mu \in (0, 1)$, we obtain

$$1 - k \ge 0, \quad \lim_{n\to\infty} k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right) > 0. \tag{21}$$

Combining (21) and Lemma 3.2, for all $p \in EP(f, C) \cap F(T)$, we have

$$\|w_n - p\| \le \|u_n - p\|. \tag{22}$$

Moreover,

$$\begin{aligned} \|u_n - p\| &= \|t_n + \theta_n(t_n - t_{n-1}) - p\| \\ &\le \|t_n - p\| + \alpha_n \cdot \frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\|. \end{aligned}$$

Since $\frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\| \to 0$ as $n \to \infty$, there exists a constant $M > 0$ such that $\frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\| \le M$. That is,

$$\|u_n - p\| \le \|t_n - p\| + \alpha_n M. \tag{23}$$

Therefore,

$$\begin{aligned} \|t_{n+1} - p\| &= \|\alpha_n g(t_n) + \beta_n t_n + (1 - \beta_n - \alpha_n)Tw_n - p\| \\ &\le \alpha_n\|g(t_n) - p\| + \beta_n\|t_n - p\| + (1 - \beta_n - \alpha_n)\|Tw_n - p\| \\ &\le \alpha_n\|g(t_n) - g(p) + g(p) - p\| + \beta_n\|t_n - p\| \\ &\quad + (1 - \beta_n - \alpha_n)\|w_n - p\| \end{aligned}$$

$$\begin{aligned}
&\le \alpha_n\rho\|t_n - p\| + \alpha_n\|g(p) - p\| + \beta_n\|t_n - p\| \\
&\quad + (1 - \beta_n - \alpha_n)\|u_n - p\| \\
&\le \alpha_n\rho\|t_n - p\| + \alpha_n\|g(p) - p\| + \beta_n\|t_n - p\| \\
&\quad + (1 - \beta_n - \alpha_n)(\|t_n - p\| + \alpha_n M) \\
&\le (1 - \alpha_n(1-\rho))\|t_n - p\| + \alpha_n\|g(p) - p\| + \alpha_n M \\
&= (1 - \alpha_n(1-\rho))\|t_n - p\| + \alpha_n(1-\rho)\frac{\|g(p) - p\| + M}{1-\rho} \\
&\le \max\left\{\|t_n - p\|, \frac{\|g(p) - p\| + M}{1-\rho}\right\} \\
&\le \cdots \le \max\left\{\|t_0 - p\|, \frac{\|g(p) - p\| + M}{1-\rho}\right\}.
\end{aligned}$$

It follows that the sequence $\{t_n\}$ is bounded. ■

Lemma 3.4: *Let $p \in EP(f, C) \cap F(T)$. Then the sequence $\{t_n\}$ generated by Algorithm 3 satisfies:*

$$d_{n+1} \le (1 - a_n)d_n + a_n b_n, \quad \forall\, n \ge 0,$$

where $d_n = \|t_n - p\|^2, a_n = \alpha_n(1-\rho), b_n = \frac{2\langle g(p)-p, t_{n+1}-p\rangle}{1-\rho} + \frac{2\|t_{n+1}-p\|}{1-\rho}\frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\|$.

Proof: In fact, for all $p \in EP(f, C) \cap F(T)$,

$$\begin{aligned}
\|u_n - p\|^2 &= \|t_n + \theta_n(t_n - t_{n-1}) - p\|^2 \\
&= \|t_n - p\|^2 + 2\theta_n\langle t_n - p, t_n - t_{n-1}\rangle + {\theta_n}^2\|t_n - t_{n-1}\|^2 \\
&\le \|t_n - p\|^2 + 2\theta_n\|t_n - p\|\|t_n - t_{n-1}\| + {\theta_n}^2\|t_n - t_{n-1}\|^2 \\
&= \|t_n - p\|^2 + \theta_n\|t_n - t_{n-1}\|\left(2\|t_n - p\| + \theta_n\|t_n - t_{n-1}\|\right) \\
&\le \|t_n - p\|^2 + \theta_n\|t_n - t_{n-1}\|M_1, \qquad (24)
\end{aligned}$$

where $M_1 := \sup_{n\in\mathbb{N}}\{2\|t_n - p\| + \theta_n\|t_n - t_{n-1}\|\}$. Furthermore, since T is nonexpansive mapping, we have $\|Tw_n - p\| \le \|w_n - p\|$. Combining (22) and (24), we get

$$\begin{aligned}
\|Tw_n - p\|^2 \le \|w_n - p\|^2 &\le \|u_n - p\|^2 \\
&\le \|t_n - p\|^2 + \theta_n\|t_n - t_{n-1}\|M_1. \qquad (25)
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\|t_{n+1}-p\|^2 &= \|\alpha_n g(t_n)+\beta_n t_n+(1-\beta_n-\alpha_n)Tw_n\|^2 \\
&= \langle \alpha_n(g(t_n)-p)+\beta_n(t_n-p) \\
&\quad +(1-\beta_n-\alpha_n)(Tw_n-p), t_{n+1}-p\rangle \\
&= \alpha_n\langle g(t_n)-g(p)+g(p)-p, t_{n+1}-p\rangle+\beta_n\langle t_n-p, t_{n+1}-p\rangle \\
&\quad +(1-\beta_n-\alpha_n)\langle Tw_n-p, t_{n+1}-p\rangle \\
&\le \alpha_n\rho\|t_n-p\|\|t_{n+1}-p\|+\alpha_n\langle g(p)-p, t_{n+1}-p\rangle \\
&\quad +\beta_n\|t_n-p\|\|t_{n+1}-p\|+(1-\beta_n-\alpha_n)\|w_n-p\|\|t_{n+1}-p\| \\
&\le (\alpha_n\rho+\beta_n)\|t_n-p\|\|t_{n+1}-p\|+\alpha_n\langle g(p)-p, t_{n+1}-p\rangle \\
&\quad +(1-\beta_n-\alpha_n)(\|t_n-p\|+\theta_n\|t_n-t_{n-1}\|)\|t_{n+1}-p\| \\
&\le [1-\alpha_n(1-\rho)]\|t_n-p\|\|t_{n+1}-p\| \\
&\quad +(1-\beta_n-\alpha_n)\theta_n\|t_n-t_{n-1}\|\|t_{n+1}-p\| \\
&\quad +\alpha_n\langle g(p)-p, t_{n+1}-p\rangle \\
&\le \frac{1}{2}[1-\alpha_n(1-\rho)][\|t_n-p\|^2+\|t_{n+1}-p\|^2] \\
&\quad +\theta_n\|t_n-t_{n-1}\|\|t_{n+1}-p\|+\alpha_n\langle g(p)-p, t_{n+1}-p\rangle \\
&\le \frac{1}{2}[1-\alpha_n(1-\rho)]\|t_n-p\|^2+\frac{1}{2}\|t_{n+1}-p\|^2 \\
&\quad +\theta_n\|t_n-t_{n-1}\|\|t_{n+1}-p\|+\alpha_n\langle g(p)-p, t_{n+1}-p\rangle. \qquad (26)
\end{aligned}
$$

That is,

$$
\begin{aligned}
\|t_{n+1}-p\|^2 &\le [1-\alpha_n(1-\rho)]\|t_n-p\|^2+2\theta_n\|t_n-t_{n-1}\|\|t_{n+1}-p\| \\
&\quad +2\alpha_n\langle g(p)-p, t_{n+1}-p\rangle \\
&= [1-\alpha_n(1-\rho)]\|t_n-p\|^2+\alpha_n(1-\rho)\left\{\frac{2\langle g(p)-p, t_{n+1}-p\rangle}{1-\rho}\right. \\
&\quad \left.+\frac{2\|t_{n+1}-p\|}{1-\rho}\frac{\theta_n}{\alpha_n}\|t_n-t_{n-1}\|\right\}.
\end{aligned}
$$

So the proof is completed. ■

Lemma 3.5: *Let $p \in EP(f, C) \cap F(T)$. Then the sequence $\{t_n\}$ generated by Algorithm 3 satisfies:*

$$
\begin{aligned}
&(1-\beta_n-\alpha_n)\left\{k\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n-v_n\|^2+k\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n-v_n\|^2\right. \\
&\quad \left.+\beta_n\|Tw_n-t_n\|^2\right\} \\
&\le \|t_n-p\|^2-\|t_{n+1}-p\|^2+\alpha_n M_2+\alpha_n(1-\beta_n-\alpha_n)\frac{\theta_n}{\alpha_n}\|t_n-t_{n-1}\|M_1,
\end{aligned}
$$

for some $M_1, M_2 > 0$.

Proof: From (3), (24), Lemma 3.2 and the definition of $\{t_{n+1}\}$, we have

$$
\begin{aligned}
\|t_{n+1} - p\|^2 &= \|\alpha_n g(t_n) + \beta_n t_n + (1-\beta_n-\alpha_n)Tw_n - p\|^2 \\
&= \|\alpha_n(g(t_n) - p) + \beta_n(t_n - p) + (1-\beta_n-\alpha_n)(Tw_n - p)\|^2 \\
&= \alpha_n\|g(t_n) - p\|^2 + \beta_n\|t_n - p\|^2 + (1-\beta_n-\alpha_n)\|Tw_n - p\|^2 \\
&\quad - \alpha_n\beta_n\|g(t_n) - t_n\|^2 - \alpha_n(1-\beta_n-\alpha_n)\|g(t_n) - Tw_n\|^2 \\
&\quad - \beta_n(1-\beta_n-\alpha_n)\|Tw_n - t_n\|^2 \\
&\le \alpha_n\|g(t_n) - p\|^2 + \beta_n\|t_n - p\|^2 + (1-\beta_n-\alpha_n)\|w_n - p\|^2 \\
&\quad - \beta_n(1-\beta_n-\alpha_n)\|Tw_n - t_n\|^2 \\
&\le \alpha_n\|g(t_n) - p\|^2 + \beta_n\|t_n - p\|^2 - \beta_n(1-\beta_n-\alpha_n)\|Tw_n - t_n\|^2 \\
&\quad + (1-\beta_n-\alpha_n)\Bigg\{\|u_n - p\|^2 - (1-k)\|u_n - w_n\|^2 \\
&\quad - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n - v_n\|^2 - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2\Bigg\} \\
&\le \alpha_n\|g(t_n) - p\|^2 + \beta_n\|t_n - p\|^2 - \beta_n(1-\beta_n-\alpha_n)\|Tw_n - t_n\|^2 \\
&\quad + (1-\beta_n-\alpha_n)\Bigg\{\|t_n - p\|^2 + \theta_n\|t_n - t_{n-1}\|M_1 \\
&\quad - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n - v_n\|^2 - k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2\Bigg\} \\
&\le \alpha_n\|g(t_n) - p\|^2 + (1-\alpha_n)\|t_n - p\|^2 \\
&\quad + (1-\beta_n-\alpha_n)\theta_n\|t_n - t_{n-1}\|M_1 \\
&\quad - (1-\beta_n-\alpha_n)\Bigg\{k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n - v_n\|^2 \\
&\quad + k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2 + \beta_n\|Tw_n - t_n\|^2\Bigg\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&(1-\beta_n-\alpha_n)\Bigg\{k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n - v_n\|^2 + k\left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n - v_n\|^2 \\
&\qquad + \beta_n\|Tw_n - t_n\|^2\Bigg\} \\
&\quad \le \|t_n - p\|^2 - \|t_{n+1} - p\|^2 + \alpha_n M_2 + \alpha_n(1-\beta_n-\alpha_n)\frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\|M_1,
\end{aligned}
$$

where $M_2 := \sup_{n\in\mathbb{N}}\{\|g(t_n) - p\|^2 - \|t_n - p\|^2\}$. ∎

Theorem 3.1: *Let $\{t_n\}$ be a sequence generated by Algorithm 3, then $\{t_n\}$ converges strongly to an element $p = P_{EP(f,C)\cap F(T)} \circ g(p)$.*

Proof: Let $p = P_{EP(f,C)\cap F(T)} \circ g(p)$, we consider the following two cases:

Case 1: There exists an $N \in \mathbb{N}$ such that $\|t_{n+1} - p\|^2 \leq \|t_n - p\|^2$ for all $n \geq N$. It follows that $\lim_{n\to\infty} \|t_n - p\|^2$ exists. First, we prove that $\lim_{n\to\infty} \|t_{n+1} - t_n\| = 0$. Indeed, applying Lemma 3.5, we can get

$$\begin{aligned}(1-\beta_n-\alpha_n)&\left\{k\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|u_n-v_n\|^2+k\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)\|w_n-v_n\|^2\right.\\&\left.+\ \beta_n\|Tw_n-t_n\|^2\right\}\\&\leq \|t_n-p\|^2-\|t_{n+1}-p\|^2+\alpha_nM_2+\alpha_n(1-\beta_n-\alpha_n)\frac{\theta_n}{\alpha_n}\|t_n-t_{n-1}\|M_1.\end{aligned} \tag{27}$$

Let $n \to \infty$, then $\alpha_n \to 0$ and $\frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\| \to 0$. In addition, from the definition of $\{\beta_n\}$, we have $\lim_{n\to\infty}(1 - \beta_n - \alpha_n) > 0$. Taking the limit of (27), we obtain

$$\lim_{n\to\infty}\|Tw_n-t_n\|=0,\quad \lim_{n\to\infty}\|u_n-v_n\|=0,\quad \lim_{n\to\infty}\|w_n-v_n\|=0. \tag{28}$$

Since $\|w_n - u_n\| \leq \|w_n - v_n\| + \|v_n - u_n\|$, we infer that

$$\lim_{n\to\infty}\|w_n-u_n\|=0. \tag{29}$$

Furthermore,

$$\begin{aligned}\|t_{n+1}-t_n\|&=\|\alpha_ng(t_n)+\beta_nt_n+(1-\beta_n-\alpha_n)Tw_n-t_n\|\\&\leq \alpha_n\|g(t_n)-t_n\|+(1-\beta_n-\alpha_n)\|Tw_n-t_n\|.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\|t_{n+1}-t_n\|=0. \tag{30}$$

Since $\{t_n\}$ is bounded, there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$ such that $t_{n_k} \rightharpoonup q$ as $k \to \infty$. Next we prove that $q \in EP(f, C) \cap F(T)$. In fact,

$$\begin{aligned}\|v_{n_k}-t_{n_k}\|&\leq \|v_{n_k}-u_{n_k}\|+\|u_{n_k}-t_{n_k}\|\\&=\|v_{n_k}-u_{n_k}\|+\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|t_{n_k}-t_{n_k-1}\|.\end{aligned}$$

Thus $\lim_{k\to\infty}\|v_{n_k} - t_{n_k}\| = 0$, we get $v_{n_k} \rightharpoonup q$. From (11) and (18), we have

$$\begin{aligned}&k\tau_{n_k}f(v_{n_k},y)\\&\quad\geq k\tau_{n_k}f(v_{n_k},w_{n_k})+\langle u_{n_k}-w_{n_k},y-w_{n_k}\rangle\end{aligned}$$

$$\geq k\left[\langle u_{n_k}-v_{n_k},w_{n_k}-v_{n_k}\rangle-\frac{\tau_{n_k}}{\tau_{n_k+1}}\frac{\mu}{2}(\|u_{n_k}-v_{n_k}\|^2+\|w_{n_k}-v_{n_k}\|^2)\right]$$
$$+\langle u_{n_k}-w_{n_k},y-w_{n_k}\rangle.$$

Since $k>0, \lim_{k\to\infty}\tau_{n_k}=\tau>0$, we obtain

$$0\leq\limsup_{k\to\infty}f(v_{n_k},y)=f(q,y),\quad \forall\, y\in C.$$

That is $q\in EP(f,C)$. Furthermore,

$$\begin{aligned}\|w_{n_k}-t_{n_k}\|&\leq\|w_{n_k}-u_{n_k}\|+\|u_{n_k}-t_{n_k}\|\\&\leq\|w_{n_k}-u_{n_k}\|+\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|t_{n_k}-t_{n_k-1}\|,\end{aligned}$$

and

$$\|Tw_{n_k}-w_{n_k}\|\leq\|Tw_{n_k}-t_{n_k}\|+\|t_{n_k}-u_{n_k}\|+\|u_{n_k}-w_{n_k}\|.$$

Taking $k\to\infty$ we have $w_{n_k}\rightharpoonup q$ and $\|Tw_{n_k}-w_{n_k}\|\to 0$. By the demiclosedness of the mapping $I-T$ and Lemma 2.2, we obtain $q\in F(T)$. Hence $q\in EP(f,C)\cap F(T)$. Combining with the definition of p, then

$$\limsup_{n\to\infty}\langle g(p)-p,t_n-p\rangle=\limsup_{k\to\infty}\langle g(p)-p,t_{n_k}-p\rangle=\langle g(p)-p,q-p\rangle\leq 0. \tag{31}$$

Therefore,

$$\begin{aligned}\limsup_{n\to\infty}\langle g(p)-p,t_{n+1}-p\rangle&\leq\limsup_{n\to\infty}\langle g(p)-p,t_{n+1}-t_n\rangle\\&\quad+\limsup_{n\to\infty}\langle g(p)-p,t_n-p\rangle\end{aligned}$$

$$= \langle g(p) - p, q - p \rangle$$
$$\leq 0. \tag{32}$$

Combining (32), Lemma 3.4 with Lemma 2.4, we conclude that $\|t_n - p\| \to 0$ as $n \to \infty$. It follows that $\{t_n\}$ converges strongly to p.

Case 2: There exists a subsequence $\{\|t_{n_j} - p\|\} \subset \{\|t_n - p\|\}$ such that $\|t_{n_j} - p\| < \|t_{n_j+1} - p\|$ for all $j \in \mathbb{N}$. From Lemma 2.3, there exists a non-decreasing sequence m_k of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and

$$\|t_{m_k} - p\| \leq \|t_{m_k+1} - p\| \quad \text{and} \quad \|t_k - p\| \leq \|t_{m_k+1} - p\|, \quad \forall\, k \in \mathbb{N}. \tag{33}$$

As proved in *Case 1*, we can get

$$\lim_{k\to\infty} \|t_{m_k+1} - t_{m_k}\| = 0.$$

Similarly, we can conclude that

$$\limsup_{k\to\infty} \langle g(p) - p, t_{m_k+1} - p \rangle \leq 0. \tag{34}$$

Applying Lemma 3.4 and (33), we have

$$\begin{aligned}
\|t_{m_k+1} - p\|^2 &\leq \left[1 - \alpha_{m_k}(1-\rho)\right] \|t_{m_k} - p\|^2 \\
&\quad + \alpha_{m_k}(1-\rho) \left\{ \frac{2\langle g(p) - p, t_{m_k+1} - p \rangle}{1-\rho} \right. \\
&\quad \left. + \frac{2\|t_{m_k+1} - p\|}{1-\rho} \frac{\theta_{m_k}}{\alpha_{m_k}} \|t_{m_k} - t_{m_k-1}\| \right\} \\
&\leq \left[1 - \alpha_{m_k}(1-\rho)\right] \|t_{m_k+1} - p\|^2 \\
&\quad + \alpha_{m_k}(1-\rho) \left\{ \frac{2\langle g(p) - p, t_{m_k+1} - p \rangle}{1-\rho} \right. \\
&\quad \left. + \frac{2\|t_{m_k+1} - p\|}{1-\rho} \frac{\theta_{m_k}}{\alpha_{m_k}} \|t_{m_k} - t_{m_k-1}\| \right\}.
\end{aligned}$$

It follows that

$$\|t_{m_k+1} - p\|^2 \leq \frac{2\langle g(p) - p, t_{m_k+1} - p \rangle}{1-\rho} + \frac{2\|t_{m_k+1} - p\|}{1-\rho} \frac{\theta_{m_k}}{\alpha_{m_k}} \|t_{m_k} - t_{m_k-1}\|. \tag{35}$$

Since $\rho \in [0, 1)$, $\lim_{k\to\infty} \frac{\theta_{m_k}}{\alpha_{m_k}} \|t_{m_k} - t_{m_k-1}\| = 0$ and $\{\|t_{m_k} - p\|\}$ is bounded. It follows from (34) and (35) that

$$\lim_{k\to\infty} \|t_{m_k+1} - p\| = 0. \tag{36}$$

Combining (36) and (33),

$$\lim_{k\to\infty} \|t_k - p\| \leq \lim_{k\to\infty} \|t_{m_k+1} - p\| = 0.$$

Hence, $t_n \to p$ as $n \to \infty$. This finishes the proof. ∎

Remark 3.2: It is easy to see that the condition $\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|t_n - t_{n-1}\| = 0$ of (C3) can be implemented easily in the numerical computation as the value of $\|t_n - t_{n-1}\|$ is known before choosing θ_n. Indeed, the parameter θ_n can be chosen such that

$$\theta_n = \begin{cases} \min\left\{\dfrac{\delta_n}{\|t_n - t_{n-1}\|}, \theta\right\}, & \text{if } t_n \neq t_{n-1}, \\ \theta, & \text{otherwise,} \end{cases}$$

where θ is a constant such that $0 < \theta < 1$ and $\{\delta_n\}$ is a positive sequence such that $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n} = 0$.

Next, on the basis of Theorem 3.1, we get a corollary of the variational inequality problem in Hilbert spaces. The classical variational inequality problem for an operator $A : \mathcal{H} \to \mathcal{H}$ is as follows: find $u^* \in C$ such that

$$\langle Au^*, v - u^*\rangle \geq 0, \quad \forall\, v \in C.$$

The solution set abbreviated as $\mathrm{VI}(C, A)$. Now, we give the following assumptions for solving the variational inequality problem:

(A1) The operator $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, i.e.

$$\langle Au, v - u\rangle \geq 0 \Longrightarrow \langle Av, u - v\rangle \leq 0, \quad \forall\, u, v \in \mathcal{H}.$$

(A2) The operator $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous if there exists a constant $L > 0$ such that

$$\|Au - Av\| \leq L\|u - v\|, \quad \forall\, u, v \in \mathcal{H}.$$

(A3) The operator $A : \mathcal{H} \to \mathcal{H}$ is sequentially weakly continuous, i.e. $\{Au_n\}$ converges weakly to Au for every sequence $\{u_n\}$ converges weakly to u.

Let $f(u, v) = \langle Au, v - u\rangle$, $\forall\, u, v \in C$, the equilibrium problem becomes the variational inequality problem with $L = 2c_1 = 2c_2$. Moreover, we have

$$v_n = \arg\min_{y\in C}\left\{\tau_n f(u_n, y) + \frac{1}{2}\|y - u_n\|^2\right\} = P_C(u_n - \tau_n Au_n),$$

where P_C is called the metric projection of $\mathcal{H}$ onto C. Therefore, we naturally get the following algorithm.

Corollary 3.1: *Assume that the feasible set C is a nonempty closed and convex subset in a real Hilbert space $\mathcal{H}$. Let $T : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping, $g : \mathcal{H} \to \mathcal{H}$ be a contraction with a constant $\rho \in [0, 1)$ and the solution set $VI(C, A) \cap F(T)$ be nonempty. Suppose that the conditions* (C1–C3) *and* (A1–A3) *hold. Then the sequence $\{t_n\}$ generated by Algorithm* 4 *converges strongly to an element $p = P_{VI(C,A)\cap F(T)} \circ g(p)$.*

Algorithm 4

Initialization: Let $\tau_1 > 0$, $\mu \in (0,1)$, $k \in (0,1]$ and $t_0, t_1 \in \mathcal{H}$.
Iterative steps: Given the current iterates t_{n-1} and t_n $(n \geq 1)$.
Step 1. Evaluate

$$v_n = P_C(u_n - \tau_n A u_n),$$

where $u_n = t_n + \theta_n(t_n - t_{n-1})$. If $v_n = u_n$, then stop. Otherwise go to **Step 2**.
Step 2. Compute

$$w_n = P_{T_n}(u_n - k\tau_n A v_n),$$

where

$$T_n := \{x \in \mathcal{H} : \langle u_n - \tau_n A u_n - v_n, x - v_n \rangle \leq 0\}.$$

Step 3. Calculate

$$t_{n+1} = \alpha_n g(t_n) + \beta_n t_n + (1 - \beta_n - \alpha_n) T w_n,$$

and

$$\tau_{n+1} = \begin{cases} \min\left\{ \dfrac{\mu(\|u_n - v_n\|^2 + \|w_n - v_n\|^2)}{2\langle A u_n - A v_n, w_n - v_n \rangle}, \tau_n \right\}, & \text{if } \langle A u_n - A v_n, w_n - v_n \rangle > 0, \\ \tau_n, & \text{otherwise.} \end{cases}$$

Set $n := n + 1$ and return back to **Step 1**.

4. Numerical experiments

In this section, we provide some numerical examples to illustrate the computational efficiency of the proposed algorithms compared to some iterative schemes in the literature. All the programs were implemented in MATLAB 2018a on a personal computer with RAM 8.00 GB. We apply the formula described in Remark 3.2 to update the inertial parameter in the related algorithms, including θ_n in Algorithm 1.

Example 4.1: Assume that the operator F is defined by

$$F(x) = \begin{pmatrix} \left(x_1^2 + (x_2 - 1)^2\right)(1 + x_2) \\ -x_1^3 - x_1 (x_2 - 1)^2 \end{pmatrix}.$$

Set $f(x,y) = \langle F(x), y - x \rangle, \forall\, x, y \in C$, where $C := \{x \in \mathbb{R}^2 : -10 \leq x_i \leq 10,\ i = 1, 2\}$. The problem EP$(f, C)$ has a unique solution $x^* = (0, -1)^{\mathsf{T}}$. Note that the operator F is pseudomonotone rather than monotone (see [58, Example 6.7]) and thus the bifunction f is pseudomonotone. We apply the proposed Algorithm 3 to solve the pseudo-monotone equilibrium problem EP(f, C).

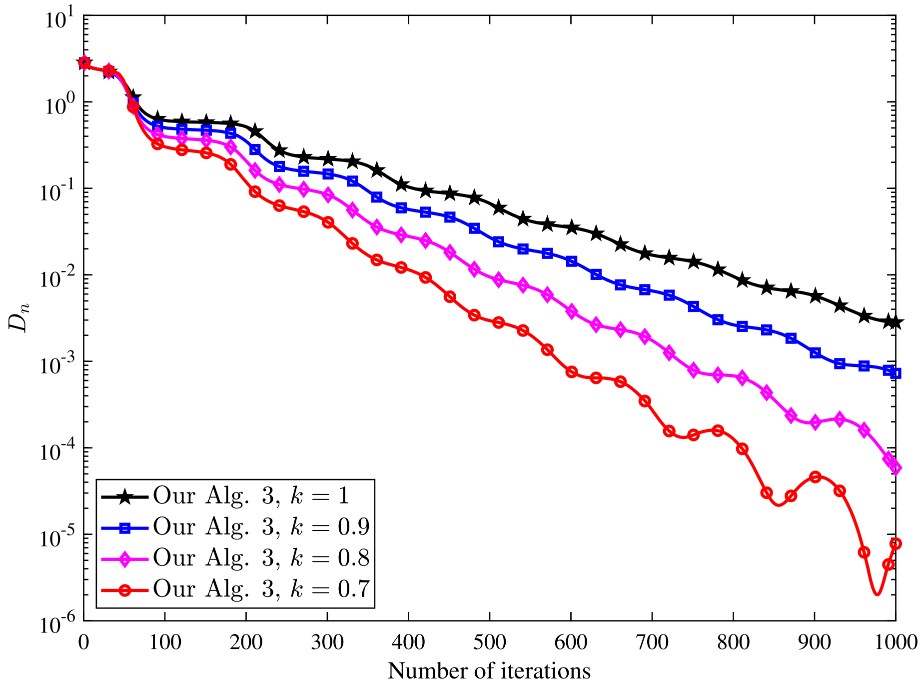

Figure 1. The numerical behavior of the proposed Algorithm 3 with different parameter k for Example 4.1.

Take $\alpha_n = 1/(10n+1)$, $\beta_n = 0.5(1-\alpha_n)$, $\theta = 0.6$, $\delta_n = 1/(10n+1)^2$, $\tau_1 = 0.1$, $\mu = 0.1$, $k = \{0.7, 0.8, 0.9, 1\}$, $g(x) = 0.1$ and $Tx = x$ for the porposed Algorithm 3. The maximum number of iterations 1000 is used as a stopping criterion. Figures 1 and 2 show the numerical behavior $D_n = \|x_n - x^*\|^2$ and the corresponding step size variations of our Algorithm 3 with different parameters k, respectively.

Example 4.2: Let the bifunction $f : C \times C \to \mathbb{R}$ be given by

$$f(x,y) = \langle Mx + Ny + c, y - x\rangle, \quad \forall\, x,y \in C,$$

where the feasible set C is defined by $C = \{x \in \mathbb{R}^m : -5 \le x_i \le 5, i = 1, 2, \ldots, m\}$, $c \in \mathbb{R}^m$ and $M, N \in \mathbb{R}^{m\times m}$. The matrix M is symmetric positive semi-definite and the matrix $(N - M)$ is symmetric negative semi-definite with Lipschitz-type constants $c_1 = c_2 = \|M - N\|/2$ (see [23, Section 6]). The matrices M, N are taken randomly (see [59, Example 5.1]). We use the proposed Algorithm 3 to solve the equilibrium problem (EP) with f and C given above, and compare it with the Algorithm 3.1 introduced by Yang and Liu [50] (shortly, YL Alg. 3.1) and the Algorithm 2.1 suggested by Shehu et al. [60] (shortly, SSTT Alg. 2.1). The parameters of all algorithms are set as follows. In all algorithms, we choose $\alpha_n = 1/(n+1)$, $\tau_1 = 0.1$, $\mu = 0.5$ and $Sx = Tx = x$. Take $\beta_n = 0.1$ for YL Alg. 3.1. Select $\alpha = 0.1$ and $\tau = 0.1$ for SSTT Alg. 2.1. Set $\theta = 0.4$, $\delta_n = 100/(n+1)^2$,

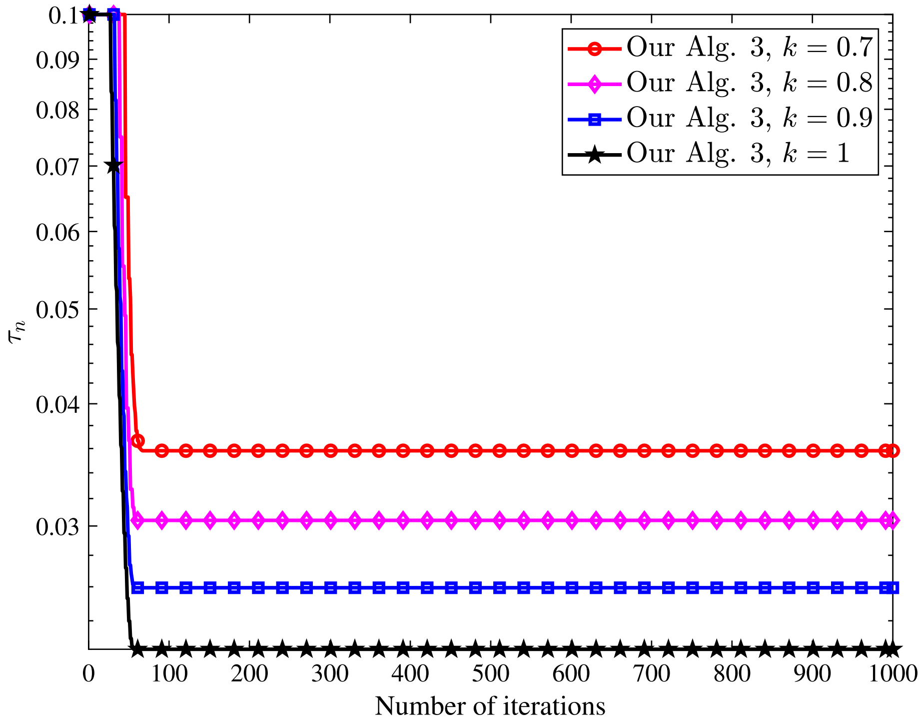

Figure 2. The step size of the proposed Algorithm 3 with different parameters k for Example 4.1.

Table 1. Numerical results of all algorithms for Example 4.2.

	$m = 5$		$m = 20$		$m = 50$		$m = 100$	
Algorithms	E_n	CPU	E_n	CPU	E_n	CPU	E_n	CPU
Our Alg. 3	2.61×10^{-08}	4.48	1.97×10^{-07}	4.03	1.09×10^{-06}	6.95	9.58×10^{-07}	12.69
YL Alg. 3.1	2.34×10^{-07}	6.26	6.02×10^{-07}	6.77	2.04×10^{-06}	9.24	4.09×10^{-06}	15.12
SSTT Alg. 2.1	1.14×10^{-07}	4.81	2.10×10^{-05}	4.55	2.79×10^{-05}	7.99	4.42×10^{-05}	14.49

$\beta_n = 0.5(1 - \alpha_n)$, $g(x) = 0.1x$ and $k = 0.8$ for the proposed Algorithm 3. Since we do not know the exact solution of the problem, the function $E_n = \|x_n - x_{n-1}\|^2$ is used to measure the computational error of all algorithms at nth step. We apply the maximum number of iterations 100 as a common stopping criterion. The numerical results of all algorithms in different dimensions are shown in Figure 3 and Table 1, where 'CPU' denotes the execution time in seconds.

Finally, we consider an example in an infinite-dimensional Hilbert space $\mathcal{H} = L^2([0,1])$ with inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t$, $\forall\, x, y \in \mathcal{H}$ and norm $\|x\| := (\int_0^1 |x(t)|^2\,\mathrm{d}t)^{1/2}$, $\forall\, x \in \mathcal{H}$.

Example 4.3: Let r, R be two positive real numbers such that $R/(d+1) < r/d < r < R$ for some $d > 1$. Assume that the feasible set C is defined by $C = \{x \in \mathcal{H} :$

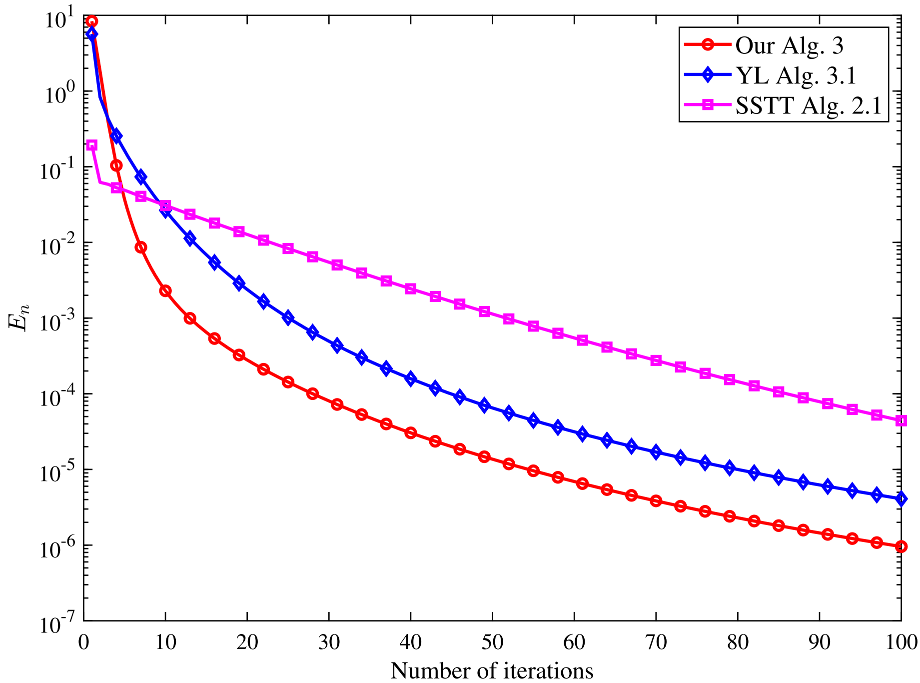

Figure 3. The numerical behavior of the proposed Algorithm 3 for Example 4.2, $m = 100$.

$\|x\| \le r\}$. The operator $A : \mathcal{H} \to \mathcal{H}$ is given by

$$A(x) = (R - \|x\|)x, \quad \forall\, x \in \mathcal{H}.$$

Note that the operator A is pseudo-monotone rather than monotone (see [61, Example 4.2]). For the experiment, we choose $R = 1.5, r = 1, d = 1.1$. The solution to the variational inequality problem with A and C given above is $x^*(t) = 0$. We compare the proposed Algorithm 4 with the Algorithms 3.1 and 3.2 introduced by Thong and Hieu [62] (shortly, TH Alg. 3.1 and TH Alg. 3.2), and the Algorithms 3.1–3.3 proposed by Tan et al. [41] (shortly, TCY Alg. 3.1, TCY Alg. 3.2, and TCY Alg. 3.3). Set $\alpha_n = 1/(n+1)$, $\mu = 0.5$, $\tau_1 = 1$, $Tx = Sx = x$ and $f(x) = g(x) = 0.1x$ for all algorithms. Choose $\beta_n = 0.5(1 - \alpha_n)$ for Our Algorithm 4, TH Alg. 3.1, TH Alg. 3.2, and TCY Alg. 3.3. Take $\theta = 0.3$, $\delta_n = 100/(n+1)^2$, and $k = 0.8$ for the suggested Algorithm 4, TCY Alg. 3.1, TCY Alg. 3.2, and TCY Alg. 3.3. Choose $\beta = 0.5$ for TCY Alg. 3.2 and TCY Alg. 3.3. Select $\xi_n = 1 + 1/(n+1)^{1.1}$ for the Algorithms 3.1–3.3 propsoed by Tan et al. [41]. The maximum number of iterations 50 is used as a common stopping criterion. We use $D_n = \|x_n(t) - x^*(t)\|^2$ to measure the error of all algorithms at the nth iteration step, and 'CPU' to denote the execution time of the algorithms in seconds. The numerical results of all algorithms with four different initial points $x_0 = x_1$ are given in Table 2.

Remark 4.1: We have the following observations for Examples 4.1–4.3.

Table 2. Numerical results of all algorithms for Example 4.3.

	$x_1 = 100t^4$		$x_1 = 100\,e^t$		$x_1 = 100\log(t)$		$x_1 = 100\sin(t)$	
Algorithms	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 4	6.36×10^{-08}	20.82	2.24×10^{-07}	23.39	1.91×10^{-07}	21.02	1.01×10^{-07}	20.93
TH Alg. 3.1	1.57×10^{-04}	19.27	5.90×10^{-04}	18.31	4.15×10^{-04}	17.86	1.89×10^{-04}	18.49
TH Alg. 3.2	4.13×10^{-03}	16.90	2.30×10^{-03}	16.67	3.85×10^{-03}	16.37	1.87×10^{-02}	16.52
TCY Alg. 3.1	3.66×10^{-17}	25.71	6.29×10^{-17}	20.68	1.12×10^{-16}	20.16	5.13×10^{-17}	21.50
TCY Alg. 3.2	5.39×10^{-19}	21.37	1.73×10^{-18}	20.84	1.76×10^{-18}	20.01	5.81×10^{-19}	21.19
TCY Alg. 3.3	7.12×10^{-23}	20.76	4.46×10^{-23}	20.21	4.47×10^{-23}	19.79	4.56×10^{-23}	21.22

- It can be seen from Figures 1 and 2 that the proposed Algorithm 3 can obtain a faster convergence speed and accuracy when choosing the appropriate parameter k.
- From the information in Figure 3 and Table 1, it can be known that the proposed Algorithm 3 has a higher accuracy and a less execution time than the algorithms presented in the literature [50,60] for the same stopping criterion, and these results are not related to the size of the dimension.
- From Table 2, it can be seen that the proposed Algorithm 4 can obtain a higher accuracy than the algorithms introduced by Thong and Hieu [62], and this result is independent of the choice of the initial values. However, our Algorithm 4 requires more execution time than the compared algorithms in [62], due to the fact that we need to spend extra time to calculate the inertial parameters in infinite-dimensional Hilbert spaces. On the other hand, our Algorithm 4 has lower accuracy and requires less computation time compared to Algorithms 3.1–3.3 proposed by Tan et al. [41] (see the numerical results for $x_1 = 100t^4$ and $x_1 = 100\sin(t)$ in Table 2). The reason for this phenomenon can be explained by the fact that the step size used by Tan et al. [41] is non-monotonic, while our Algorithm 4 uses a non-increasing step size. In our future work we will consider using this non-monotonic type of step size to speed up the convergence of the algorithm.

5. Conclusions

In this paper, we propose a new inertial self-adaptive subgradient extragradient method for solving pseudomonotone equilibrium problems and fixed point problems in Hilbert spaces. Under suitable conditions, the sequence generated by the proposed algorithm strongly converges to the common solution of equilibrium problems and fixed point problems. Particularly, we add a new variable k that can control the step size, and it is reflected in Figures 1 and 2 that this variable can effectively accelerate the convergence rate. In addition, we apply our main results to deal with the variational inequality problem. By comparing with other related results [50,60,62], our algorithms do have a better convergence effects.

Disclosure statement

No potential conflict of interest was reported by the author(s).

Funding

This work was supported by the NSF of China [grant number 12171062] and the Natural Science Foundation of Chongqing [grant number CSTB2022NSCQ-JQX0004]. This work was supported by National Natural Science Foundation of China [11771063].

ORCID

Bing Tan http://orcid.org/0000-0003-1509-1809

References

[1] Fan K. A minimax inequality and applications. In: Inequalities III. New York: Academic Press; 1972. p. 103–113.
[2] Ceng LC, Ansari QH, Yao JC. Viscosity approximation methods for generalized equilibrium problems and fixed point problems. J Global Optim. 2009;43:487–502.
[3] Ceng LC, Guu SM, Hu HY, et al. Hybrid shrinking projection method for a generalized equilibrium problem, a maximal monotone operator and a countable family of relatively nonexpansive mappings. Comput Math Appl. 2011;61:2468–2479.
[4] Ceng LC, Latif A, Al-Mazrooei AE. Hybrid viscosity methods for equilibrium problems, variational inequalities, and fixed point problems. Appl Anal. 2016;95:1088–1117.
[5] Dong QL, Yao Y. An iterative method for finding common solutions of system of equilibrium problems and fixed point problems in Hilbert spaces. An Stiint Univ 'Ovidius' Constanta Ser Mat. 2011;19:101–116.
[6] Facchinei F, Kanzow C. Penalty methods for the solution of generalized Nash equilibrium problems. SIAM J Optim. 2010;20:2228–2253.
[7] Korpelevich GM. The extragradient method for finding saddle points and other problems. Ekonomikai Matematicheskie Metody. 1976;12:747–756.
[8] Kumam P, Jitpeera T. Iterative algorithms for mixed equilibrium problems, system of quasi-variational inclusion, and fixed point problem in Hilbert spaces. Abstr Appl Anal. 2014;2014:1–17.
[9] Kumam P, Katchang P. The hybrid algorithm for the system of mixed equilibrium problems, the general system of finite variational inequalities and common fixed points for nonexpansive semigroups and strictly pseudo-contractive mappings. Fixed Point Theory Appl. 2012;2012:Article ID 84.
[10] Kumam W, Piri H, Kumam P. Solutions of system of equilibrium and variational inequality problems on fixed points of infinite family of nonexpansive mappings. Appl Math Comput. 2014;248:441–455.
[11] Parthasarathy T, Raghavan TES. Existence of saddle points and Nash equilibrium points for differential games. SIAM J Control. 1975;13:977–980.
[12] Shehu Y, Collins CO. Fixed points solutions of generalized equilibrium problems and variational inequality problems. J Appl Math Inform. 2010;28:1263–1275.
[13] Shehu Y. Convergence theorems by hybrid method for systems of equilibrium problems and fixed point problem. Math Comput Model. 2011;54:1943–1953.
[14] Shehu Y, Ogbuisi FU. An iterative algorithm for finding a common solution of fixed point problem and system of equilibrium problems. Dyn Contin Discrete Impuls Syst Ser B Appl Algorithms. 2016;23:251–267.

[15] Tan B, Qin X, Cho SY. Revisiting subgradient extragradient methods for solving variational inequalities. Numer Algorithms. 2022;90:1593–1615.

[16] Yao Y, Petruel A, Postolache M. Iterative computation of mixed equilibrium problems and fixed point problems. J Nonlinear Convex Anal. 2018;19:2099–2108.

[17] Sunthrayuth P, Jolaoso LO, Cholamjiak P. New Bregman projection methods for solving pseudo-monotone variational inequality problem. J Appl Math Comput. 2021. DOI:10.1007/s12190-021-01581-2

[18] Moudafi A. Proximal point algorithm extended to equilibrium problems. J Nat Geom. 1999;15:91–100.

[19] Rockafellar RT. Monotone operators and the proximal point algorithm. SIAM J Control Optim. 1976;14:877–898.

[20] Mastroeni G. On auxiliary principle for equilibrium problems. In: Daniele P, et al., editors. Equilibrium problems and variational models. Based on the Meeting; Erice, Italy; 2000 Jun 23–Jul 2. Boston (MA): Kluwer Academic Publishers; 2003. p. 289–298. (Nonconvex optim appl.; 68).

[21] Martinet B. Régularisation d'inéquations variationnelles par approximations successives. Rev Francaise Informat Recherche Opérationnelle. 1970;4:154–158.

[22] Flam SD, Antipin AS. Equilibrium programming using proximal-like algorithms. Math Program. 1997;78:29–41.

[23] Tran DQ, Dung ML, Nguyen VH. Extragradient algorithms extended to equilibrium problems. Optimization. 2008;57:749–776.

[24] Hieu DV, Cho YJ, Xiao Y. Modified extragradient algorithms for solving equilibrium problems. Optimization. 2018;67:2003–2029.

[25] Iusem AN, Mohebbi V. Extragradient methods for nonsmooth equilibrium problems in Banach spaces. Optimization. 2020;69:2383–2403.

[26] Iusem AN, Mohebbi V. Extragradient methods for vector equilibrium problems in Banach spaces. Numer Funct Anal Optim. 2019;40:993–1022.

[27] Kumam P. A relaxed extragradient approximation method of two inverse-strongly monotone mappings for a general system of variational inequalities, fixed point and equilibrium problems. Bull Iranian Math Soc. 2010;36:227–250.

[28] Kumam P, Katchang P. A viscosity of extragradient approximation method for finding equilibrium problems, variational inequalities and fixed point problems for nonexpansive mappings. Nonlinear Anal Hybrid Syst. 2009;3:475–486.

[29] Rehman HU, Kumam P, Dong QL, et al. A modified self-adaptive extragradient method for pseudomonotone equilibrium problem in a real Hilbert space with applications. Math Methods Appl Sci. 2021;44:3527–3547.

[30] Thong DV, Gibali A. Extragradient methods for solving non-Lipschitzian pseudo-monotone variational inequalities. J Fixed Point Theory Appl. 2019;21:19.

[31] Thong DV, Hieu DV. Strong convergence of extragradient methods with a new step size for solving variational inequality problems. Comput Appl Math. 2019;38:21.

[32] Thong DV, Triet NA, Li XH, et al. Strong convergence of extragradient methods for solving bilevel pseudo-monotone variational inequality problems. Numer Algorithms. 2020;83:1123–1143.

[33] Yao Y, Noor MA, Noor KI, et al. Modified extragradient methods for a system of variational inequalities in Banach spaces. Acta Appl Math. 2010;110:1211–1224.

[34] Yao Y, Postolache M, Yao JC. Strong convergence of an extragradient algorithm for variational inequality and fixed point problems. Politehn Univ Bucharest Sci Bull Ser A Appl Math Phys. 2020;82:3–12.

[35] Bot RI, Csetnek ER. A hybrid proximal-extragradient algorithm with inertial effects. Numer Funct Anal Optim. 2015;36:951–963.

[36] Dong QL, Cho YJ, Zhong LL, et al. Inertial projection and contraction algorithms for variational inequalities. J Global Optim. 2018;70:687–704.
[37] Iyiola OS, Ogbuisi FU, Shehu Y. An inertial type iterative method with Armijo linesearch for nonmonotone equilibrium problems. Calcolo. 2018;55:22.
[38] Qin X, Wang L, Yao JC. Inertial splitting method for maximal monotone mappings. J Nonlinear Convex Anal. 2020;21:2325–2333.
[39] Shehu Y, Gibali A, Sagratella S. Inertial projection-type methods for solving quasi-variational inequalities in real Hilbert spaces. J Optim Theory Appl. 2020;184:877–894.
[40] Tan B, Qin X, Yao JC. Strong convergence of self-adaptive inertial algorithms for solving split variational inclusion problems with applications. J Sci Comput. 2021;87:Article ID 20.
[41] Tan B, Cho SY, Yao JC. Accelerated inertial subgradient extragradient algorithms with non-monotonic step sizes for equilibrium problems and fixed point problems. J Nonlinear Var Anal. 2022;6:89–122.
[42] Thong DV, Hieu DV. Inertial extragradient algorithms for strongly pseudomonotone variational inequalities. J Comput Appl Math. 2018;341:80–98.
[43] Yao Y, Qin X, Yao JC. Convergence analysis of an inertial iterate for the proximal split feasibility problem. J Nonlinear Convex Anal. 2019;20:489–498.
[44] Suantai S, Pholasa N, Cholamjiak P. The modified inertial relaxed CQ algorithm for solving the split feasibility problems. J Ind Manag Optim. 2018;14:1595–1615.
[45] Suantai S, Pholasa N, Cholamjiak P. Relaxed CQ algorithms involving the inertial technique for multiple-sets split feasibility problems. Rev R Acad Cienc Exactas Fís Nat Ser A Mat RACSAM. 2019;113:1081–1099.
[46] Cholamjiak P, Thong DV, Cho YJ. A novel inertial projection and contraction method for solving pseudomonotone variational inequality problems. Acta Appl Math. 2020;169:217–245.
[47] Cai G, Bu S. Strong and weak convergence theorems for general mixed equilibrium problems and variational inequality problems and fixed point problems in Hilbert spaces. J Comput Appl Math. 2013;247:34–52.
[48] Qin X, Cho SY, Kang SM, et al. On the hybrid projection method for fixed point and equilibrium problems. TOP. 2013;21:341–354.
[49] Qin X, Cho YJ, Kang SM. Convergence theorems of common elements for equilibrium problems and fixed point problems in Banach spaces. J Comput Appl Math. 2009;225:20–30.
[50] Yang J, Liu H. The subgradient extragradient method extended to pseudomonotone equilibrium problems and fixed point problems in Hilbert space. Optim Lett. 2020;14:1803–1816.
[51] Yang P, Yao Y. On modified iterative method for generalized equilibrium problems and fixed point problems. J Adv Math Stud. 2011;4:97–108.
[52] Yao Y, Liou YC, Wong MM. Algorithms for equilibrium problems and fixed point problems approach to minimization problems. Taiwanese J Math. 2010;14:2073–2089.
[53] Censor Y, Gibali A, Reich S. The subgradient extragradient method for solving variational inequalities in Hilbert space. J Optim Theory Appl. 2011;148:318–335.
[54] Peypouquet J. Convex optimization in normed spaces: theory, methods and examples. Berlin: Springer; 2015.
[55] Jung JS. Iterative approaches to common fixed points of nonexpansive mappings in Banach spaces. J Math Anal Appl. 2005;302:509–520.
[56] Maingé PE. Strong convergence of projected subgradient methods for nonsmooth and nonstrictly convex minimization. Set-Valued Anal. 2008;16:899–912.

[57] Xu HK. Iterative algorithms for nonlinear operators. J London Math Soc. 2002;66:240–256.
[58] Shehu Y, Dong QL, Jiang D. Single projection method for pseudo-monotone variational inequality in Hilbert spaces. Optimization. 2019;68:385–409.
[59] Rehman HU, Kumam P, Gibali A, et al. Convergence analysis of a general inertial projection-type method for solving pseudomonotone equilibrium problems with applications. J Inequal Appl. 2021;2021:Article ID 63.
[60] Shehu Y, Iyiola OS, Thong DV, et al. An inertial subgradient extragradient algorithm extended to pseudomonotone equilibrium problems. Math Methods Oper Res. 2020;93:213–242.
[61] Tan B, Liu L, Qin X. Self adaptive inertial extragradient algorithms for solving bilevel pseudomonotone variational inequality problems. Jpn J Ind Appl Math. 2021;38:519–543.
[62] Thong DV, Hieu DV. Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems. Numer Algorithms. 2019;82:761–789.