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INERTIAL HYBRID AND SHRINKING PROJECTION ALGORITHMS FOR SOLVING VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we propose two inertial hybrid and shrinking projection algorithms for strict pseudo-contractions in Hilbert spaces and obtain strong theorems in general conditions. In addition, we also propose two new inertial hybrid and shrinking projection algorithms without extrapolating step for nonexpansive mappings in Hilbert spaces and get strong convergence results. Finally, we give some numerical examples to illustrate the computational performance of our proposed algorithms.

1. Introduction

Let C be a nonempty convex closed subset in a real Hilbert space H. For all $x, y \in C$, a mapping $T : C \to C$ is said to be (i) Lipschitzian if $\|Tx-Ty\| \leq L\|x-y\|$ for some $L > 0$; (ii) nonexpansive if $L = 1$; (iii) τ-strict pseudo-contraction if there exists a constant $0 \leq \tau < 1$ such that $\|Tx-Ty\|^2 \leq \|x-y\|^2+\tau\|(I-T)x-(I-T)y\|^2$; (iv) γ-inverse strongly monotone if there exists a positive constant γ such that $\langle x - y, Tx - Ty\rangle \geq \gamma\|Tx - Ty\|^2$. The set of fixed points of a mapping $T : C \to C$ is defined by $\mathrm{Fix}(T) := \{x \in C : Tx = x\}$. Approximation of fixed point problems with nonexpansive mappings and strict pseudo-contractions have various specific applications, because many problems can be considered as fixed point problems with nonexpansive mappings and strict pseudo-contractions. For instance, monotone variational inequalities, convex optimization problems, convex feasibility problems and image restoration problems; see, e.g., [2, 3, 7, 24, 26, 33]. It is known that the Mann's iteration method $x_{n+1} = \delta_n x_n + (1 - \delta_n) Tx_n$, $n \geq 0$ converges weakly to a fixed point of T provided that $\{\delta_n\} \subset (0, 1)$ satisfies $\sum_{n=1}^{\infty} \delta_n (1 - \delta_n) = +\infty$. In 1967, Browder and Petryshyn [4] obtained the first convergence result for τ-strict pseudo-contraction in Hilbert spaces. However, iterative algorithms for strict pseudo-contractions are far less developed than those for nonexpansive mappings. Strict pseudo-contractions have many applications due to their connection with inverse strong monotone operators. Indeed, if A is a τ-strict pseudo-contraction, then $T = I - A$ is a $\frac{1-\tau}{2}$-inverse strongly monotone. So we can restate the problem of zeros for A to the fixed point problem of T, and vice versa. It is known that Mann algorithm has weak convergence in the wider setting of strict pseudo-contractions mappings. In practical applications, many problems, such as, quantum physics and

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image reconstruction, are in infinite dimensional spaces. To investigate these problems, norm convergence is usually preferable to the weak convergence. Therefore, modifying the Mann iteration method to obtain strong convergence is an important research topic; see, e.g., [1, 8, 9, 11, 12] and the references therein. In this paper, we mainly focus on the projection type algorithms. Let us review some classic results. In 2003, Nakajo and Takahashi established strong convergence of the Mann iteration with the aid of projections, see [22]. This method is now referred as the hybrid projection method. Inspired by Nakajo and Takahashi [22], Takahashi, Takeuchi and Kubota [28] also proposed a projection-based method and obtained strong convergence results, which is now called the shrinking projection method. In recent years, many authors studied these projection-based methods in various spaces and problems, see, for instance [10, 27, 29, 30] and the references therein. In general, the convergence rate of Mann algorithm is slow. Fast convergence of algorithm is required in many practical applications. In particular, an inertial type extrapolation was first proposed by Polyak [23] as an acceleration process. In recent years, some authors have constructed different fast iterative algorithms by inertial extrapolation techniques; see, e.g., [5, 15, 17, 18, 25, 31]. Recently, based on the projection method and the hybrid method, Malitsky and Semenov [19] introduced a new hybrid method without extrapolation step for solving variational inequality problems, and proved a strong convergence theorem. Their numerical experiments show that this method has a competitive performance.

Inspired and motivated by the above works, in this paper, based on inertial ideas and the projection type algorithms, we propose two inertial hybrid and shrinking projection algorithms for strict pseudo-contractions in Hilbert spaces and analyze the convergence of the proposed algorithms. In addition, we also propose two new inertial hybrid and shrinking projection algorithms without extrapolating step for nonexpansive mappings in Hilbert spaces. This paper is organized as follows. Section 2 gives the mathematical preliminaries. Section 3 presents two algorithms for strict pseudo-contractions and analyzes their convergence. Section 4 proposes two new algorithms without extrapolating step for nonexpansive mapping and analyzes their convergence. Finally, in the last section, we provide some numerical experiments to illustrate the convergence behavior of the proposed algorithms.

2. Preliminaries

Throughout this paper, we denote the strong and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. Let $\omega_w\{x_n\}$ denote the set of all weak limits of $\{x_n\}$. There exists a unique nearest point in C, denoted by $P_C x$, such that $P_C(x) := \operatorname{argmin}_{y\in C}\|x-y\|$ for any $x \in H$, where P_C is called the metric projection of H onto C. $P_C x$ is characterized by the properties

$$P_C x \in C \quad \text{and} \quad \langle P_C x - x, P_C x - y\rangle \le 0, \quad \forall y \in C. \tag{2.1}$$

This characterization implies the following inequality

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \le \|x-y\|^2, \quad \forall x \in H, \forall y \in C. \tag{2.2}$$

We give some special cases with simple analytical solutions:

(i) The Euclidean projection of x_0 onto a halfspace $H_{a,b}^{-} = \{x : \langle a, x\rangle \leq b\}$ is given by $P_{H_{a,b}^{-}} x = x - \frac{[\langle a,x\rangle - b]_+}{\|a\|^2} a$.

(ii) The Euclidean projection of x_0 onto an Euclidean ball $B[c, r] = \{x : \|x - c\| \leq r\}$ is given by $P_{B[c,r]} x = c + \frac{r}{\max\{\|x-c\|, r\}}(x - c)$.

(iii) The Euclidean projection of x_0 onto a box $\text{Box}[\ell, u] = \{x : \ell \leq x \leq u\}$ is given by $P_{\text{Box}[\ell,u]} x_i = \min\{\max\{x_i, \ell_i\}, u_i\}$.

Lemma 2.1 ([6])**.** *Let C be a nonempty closed convex subset of a real Hilbert space H, $T : C \to H$ be a nonexpansive mapping. Let $\{x_n\}$ be a sequence in C and $x \in H$ such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to +\infty$. Then $x \in \text{Fix}(T)$.*

Lemma 2.2 ([16])**.** *Let C be a nonempty closed and convex subset of a real Hilbert space H. Given $x, y, z \in H$ and $a \in R$. $\{v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Lemma 2.3 ([20])**.** *Let C be a closed convex subset of H, $\{x_n\} \subset H$ and $u \in H$. Let $q = P_C u$. If $\omega_w\{x_n\} \subset C$ and satisfies the condition $\|x_n - u\| \leq \|u - q\|$, $\forall n \in N$. Then $x_n \to q$.*

Lemma 2.4 ([13])**.** *Let $\{a_n\}$ and $\{\xi_n\}$ be nonnegative real sequences, $\alpha \in [0, 1)$, $\beta \in R^+$. For all $n \in N$ the following inequality holds: $a_{n+1} \leq \alpha a_n + \beta\xi_n$, $\forall n \geq 1$. If $\sum_{n=1}^{\infty} \xi_n < +\infty$, then $\lim_{n\to\infty} a_n = 0$.*

Proposition 2.5 ([21])**.** *Assume that C is a closed convex subset of a Hilbert space H. Let $T : C \to C$ be a self-mapping of C.*

(i) *If T is a τ-strict pseudo-contraction, then T satisfies the Lipschitz condition*

$$\|Tx - Ty\| \leq \frac{1+\tau}{1-\tau}\|x - y\|, \quad \forall x, y \in C. \tag{2.3}$$

(ii) *If T is a τ-strict pseudo-contraction, then the mapping $I - T$ is demiclosed (at 0). That is, if $\{x_n\}$ is a sequence in C such that $x_n \to x^*$ and $(I - T)x_n \to 0$, then $(I - T)x^* = 0$.*

(iiii) *If T is a τ-strict pseudo-contraction, then $\text{Fix}(T)$ is closed and convex so that the projection $P_{\text{Fix}}(T)$ is well defined.*

3. Inertial hybrid and shrinking projection algorithms

In this section, by combining the inertial extrapolation with the hybrid method and the shrinking method, respectively, we introduce two inertial hybrid and shrinking projection algorithms for strict pseudo-contractions in Hilbert spaces and analyze their convergence.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a τ-strict pseudo-contraction for some $0 \leq \tau < 1$ such that $\text{Fix}(T) \neq \emptyset$. Let*

$$\delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset (0, 1). \tag{3.1}$$

Set $x_{-1}, x_0 \in C$ arbitrarily. Define a sequence $\{x_n\}$ by the following algorithm:

$$\left\{\begin{array}{l} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ y_n = \psi_n w_n + (1 - \psi_n) T w_n, \\ C_n = \Big\{ z \in C : \|y_n - z\|^2 \leq \|w_n - z\|^2 \\ \qquad\qquad + (1 - \psi_n)(\tau - \psi_n) \|w_n - T w_n\|^2 \Big\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geq 0. \end{array}\right. \tag{3.2}$$

Then the iterative sequence $\{x_n\}$ defined by (3.2) *converges strongly to $P_{\mathrm{Fix}(T)} x_0$.*

Proof. Our proof is divided into three steps.
Step 1. We show that $\mathrm{Fix}(T) \subset C_n \cap Q_n$. First observe that C_n is convex by Lemma 2.2. Next we show that $\mathrm{Fix}(T) \subset C_n$ for all $n \geq 0$. Indeed, for all $u \in \mathrm{Fix}(T)$, we have

$$\begin{aligned} \|y_n - u\|^2 &\leq \psi_n \|w_n - u\|^2 + (1 - \psi_n) \left(\|w_n - u\|^2 + \tau \|w_n - T w_n\|^2 \right) \\ &\quad - \psi_n (1 - \psi_n) \|w_n - T w_n\|^2 \\ &= \|w_n - u\|^2 + (1 - \psi_n)(\tau - \psi_n) \|w_n - T w_n\|^2 . \end{aligned}$$

Thus $u \in C_n$ for all $n \geq 0$. For $n = 0$, we have $\mathrm{Fix}(T) \subset C = Q_0$. Assume that $\mathrm{Fix}(T) \subset Q_{n-1}$, combining the fact that $x_n = P_{C_{n-1} \cap Q_{n-1}} x_0$ and (2.1), we obtain $\langle x_n - z, x_n - x_0 \rangle \leq 0$, $\forall z \in C_{n-1} \cap Q_{n-1}$. As $\mathrm{Fix}(T) \subset C_{n-1} \cap Q_{n-1}$ by the induction assumption, this together with the definition of Q_n implies that $\mathrm{Fix}(T) \subset Q_n$ and hence $\mathrm{Fix}(T) \subset C_n \cap Q_n$ for all $n \geq 0$.
Step 2. We show that $\|x_{n+1} - x_n\| \to 0$. From the definition of Q_n and $\mathrm{Fix}(T) \subset Q_n$, we have $\|x_n - x_0\| \leq \|u - x_0\|$, for all $u \in \mathrm{Fix}(T)$. In particular, $\{x_n\}$ is bounded and

$$\|x_n - x_0\| \leq \|x^* - x_0\| , \quad \text{where } x^* = P_{\mathrm{Fix}(T)} x_0. \tag{3.3}$$

From the fact that $x_{n+1} \in Q_n$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$. Using (2.2), we have

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 . \tag{3.4}$$

Therefore, combining (3.3) and (3.4) we obtain

$$\begin{aligned} \sum_{n=1}^{N} \|x_{n+1} - x_n\|^2 &\leq \sum_{n=1}^{N} \left(\|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \right) \\ &\leq \|x^* - x_0\|^2 - \|x_1 - x_0\|^2 , \end{aligned}$$

which implies that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent. Therefore,

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.5}$$

Next, from (3.1), (3.5) and the definition of w_n, we get

$$\|x_n - w_n\| = \delta_n \|x_n - x_{n-1}\| \leq \delta_2 \|x_n - x_{n-1}\| \to 0. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\|x_{n+1} - w_n\| \leq \|x_{n+1} - x_n\| + \|x_n - w_n\| \to 0. \tag{3.7}$$

Step 3. We show that $x_n \to x^*$, where $x^* = P_{\mathrm{Fix}(T)} x_0$. By the fact $x_{n+1} \in C_n$ we get

$$\|x_{n+1} - y_n\|^2 \le \|x_{n+1} - w_n\|^2 + (1-\psi_n)(\tau - \psi_n)\|w_n - Tw_n\|^2 . \tag{3.8}$$

Moreover, by the definition of y_n, we get that

$$\begin{aligned} \|x_{n+1} - y_n\|^2 = {} & \psi_n \|x_{n+1} - w_n\|^2 + (1-\psi_n)\|x_{n+1} - Tw_n\|^2 \\ & - \psi_n (1-\psi_n)\|w_n - Tw_n\|^2 . \end{aligned} \tag{3.9}$$

Combining (3.1), (3.8) and (3.9), we deduce that

$$\|x_{n+1} - Tw_n\|^2 \le \|x_{n+1} - w_n\|^2 + \tau \|w_n - Tw_n\|^2 . \tag{3.10}$$

On the other hand, we have

$$\|x_{n+1} - Tw_n\|^2 = \|x_{n+1} - w_n\|^2 + \|w_n - Tw_n\|^2 + 2\langle x_{n+1} - w_n, w_n - Tw_n\rangle . \tag{3.11}$$

Combining (3.10) and (3.11) we obtain

$$(1-\tau)\|w_n - Tw_n\|^2 \le -2\langle x_{n+1} - w_n, w_n - Tw_n\rangle . \tag{3.12}$$

It follows from (3.7) and (3.12), we obtain

$$\|w_n - Tw_n\| \le \frac{2}{1-\tau}\|x_{n+1} - w_n\| \to 0. \tag{3.13}$$

On the other hand, by (2.3) and (3.6), we have

$$\|Tx_n - Tw_n\| \le \frac{1+\tau}{1-\tau}\|x_n - w_n\| \to 0. \tag{3.14}$$

Therefore, combining (3.6), (3.13) and (3.14), we obtain

$$\|Tx_n - x_n\| \le \|Tx_n - Tw_n\| + \|Tw_n - w_n\| + \|w_n - x_n\| \to 0. \tag{3.15}$$

By (3.15) and Proposition 2.5 (ii), it follows that every weak limit point of $\{x_n\}$ is a fixed point of T, i.e., $\omega_w\{x_n\} \subset \mathrm{Fix}(T)$. This fact, with the inequality (3.3) and Lemma 2.3, ensures the strong convergence of $\{x_n\}$ to $P_{\mathrm{Fix}(T)} x_0$. This completes the proof. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a τ-strict pseudo-contraction for some $0 \le \tau < 1$ such that* $\mathrm{Fix}(T) \neq \emptyset$. *Let*

$$\delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset (0,1). \tag{3.16}$$

Set $x_{-1}, x_0 \in C$ arbitrarily. Define a sequence $\{x_n\}$ by the following algorithm:

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ y_n = \psi_n w_n + (1-\psi_n) Tw_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_n - z\|^2 \le \|w_n - z\|^2 \\ \qquad\qquad + (1-\psi_n)(\tau - \psi_n)\|w_n - Tw_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, n \ge 0. \end{cases} \tag{3.17}$$

Then the iterative sequence $\{x_n\}$ defined by (3.17) *converges strongly to* $P_{\mathrm{Fix}(T)} x_0$.

Proof. Our proof is divided into three steps.
Step 1. We show that $\mathrm{Fix}(T) \subset C_{n+1}$ for all $n \geq 0$. According to Step 1 in Theorem 3.1, for all $u \in \mathrm{Fix}(T)$, we have $\|y_n - u\|^2 \leq \|w_n - u\|^2 + (1-\psi_n)(\tau - \psi_n)\|w_n - Tw_n\|^2$. So $u \in C_{n+1}$ for each $n \geq 0$ and thus $\mathrm{Fix}(T) \subset C_{n+1}$.
Step 2. We show that $\|x_{n+1} - x_n\| \to 0$. From $x_n = P_{C_n} x_0$, this together with the fact $\mathrm{Fix}(T) \subset C_n$ further implies $\|x_n - x_0\| \leq \|u - x_0\|$, for all $u \in \mathrm{Fix}(T)$. In particular, $\{x_n\}$ is bounded and $\|x_n - x_0\| \leq \|x^* - x_0\|$, where $x^* = P_{\mathrm{Fix}(T)} x_0$. The fact $x_{n+1} \in C_{n+1} \subset C_n$, we have $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, this implies that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Using (2.2), we have $\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2, \quad \forall n \geq 0$. This implies that $\|x_{n+1} - x_n\| \to 0$. Also, we have $\|x_n - w_n\| \to 0$ and $\|x_{n+1} - w_n\| \to 0$.
Step 3. We show that $x_n \to x^*$, where $x^* = P_{\mathrm{Fix}(T)} x_0$. This result can be easily proved by using a similar way as Step 3 in Theorem 3.1. We leave the proof for the reader to verify. □

Remark 3.3. (i) We know that τ-strict pseudo-contraction contains nonexpansive. In fact, 0-strict pseudo-contraction is nonexpansive. Recently, Dong et al. [14] introduced an inertial hybrid projection algorithm for nonexpansive mappings.
(ii) When $\delta_n = 0$, the Algorithm (3.2) transformed into a hybrid projection algorithm for strict pseudo-contraction introduced by Marino and Xu [21]. When $\delta_n = 0$ and T is nonexpansive, the Algorithm (3.2) transformed into a hybrid projection algorithm proposed by Nakajo and Takahashi [22], the Algorithm (3.17) transformed into a shrinking projection algorithm proposed by Takahashi, Takeuchi and Kubota [28].
(iii) The conditions (3.1) on $\{\delta_n\}$ and $\{\psi_n\}$ in the Algorithm (3.2) are obviously relaxed. To the best of our knowledge, the conditions of convergence of the Algorithm (3.2) are the weakest among the inertial algorithms.

4. Inertial hybrid and Shrinking projection algorithms without extrapolating step

In this section, we introduce two inertial hybrid and shrinking projection algorithms without extrapolating step for nonexpansive mapping in Hilbert spaces and analyze their convergence.

Theorem 4.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let*

$$\delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset \left(\frac{\sigma}{1+\sigma}, 1\right), \sigma \in (0,1). \tag{4.1}$$

Set $x_{-1}, x_0, y_0 \in C$ arbitrarily. Define two sequences $\{x_n\}$ and $\{y_n\}$ by the following algorithm:

$$\left\{\begin{array}{l} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ y_{n+1} = \psi_n w_n + (1-\psi_n) T y_n, \\ C_n = \Big\{ z \in C : \|y_{n+1} - z\|^2 \leq \psi_n \|w_n - z\|^2 + (1-\psi_n) \|y_n - z\|^2 \\ \qquad\qquad\qquad\qquad -\psi_n (1-\psi_n) \|w_n - T y_n\|^2 \Big\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geq 0. \end{array}\right. \tag{4.2}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (4.2) *converge strongly to $P_{\mathrm{Fix}(T)} x_0$.*

Proof. First observe that C_n is convex by Lemma 2.2. Next we show that $\mathrm{Fix}(T) \subset C_n$ for all $n \geq 0$. Indeed, for all $u \in \mathrm{Fix}(T)$, we have

$$\|y_{n+1} - u\|^2 \leq \psi_n \|w_n - u\|^2 + (1-\psi_n) \|y_n - u\|^2 - \psi_n (1-\psi_n) \|w_n - T y_n\|^2 .$$

Thus $u \in C_n$ for all $n \geq 0$. Using a similar way in Theorem 3.1, we can get that $\mathrm{Fix}(T) \subset C_n \cap Q_n$ for all $n \geq 0$. Furthermore, we can prove that

$$\|x_n - x_0\| \leq \|x^* - x_0\|, \quad \text{where } x^* = P_{\mathrm{Fix}(T)} x_0, \tag{4.3}$$

and

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty, \quad \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0, \tag{4.4}$$

and

$$\lim_{n\to\infty} \|w_n - x_n\| = 0, \quad \lim_{n\to\infty} \|w_n - x_{n+1}\| = 0. \tag{4.5}$$

On the other hand, by the definition of w_n in (4.2), we have

$$\begin{aligned} \|w_n - x_{n+1}\|^2 &\leq \|x_n - x_{n+1}\|^2 + \delta_n^2 \|x_n - x_{n-1}\| \\ &\quad + \delta_n \left[\|x_n - x_{n+1}\|^2 + \|x_n - x_{n-1}\|^2 \right] \\ &\leq (1+\delta_n) \|x_n - x_{n+1}\|^2 + \delta_n (1+\delta_n) \|x_n - x_{n-1}\|^2 . \end{aligned} \tag{4.6}$$

Combining (4.1), (4.2), (4.4), (4.6) and the fact $x_{n+1} \in C_n$, we obtain

$$\begin{aligned} \|y_{n+1} - x_{n+1}\|^2 \leq& \psi_n \|w_n - x_{n+1}\|^2 + (1-\psi_n) \|y_n - x_{n+1}\|^2 \\ \leq& \psi_n \left[(1+\delta_n) \|x_n - x_{n+1}\|^2 + \delta_n (1+\delta_n) \|x_n - x_{n-1}\|^2 \right] \\ &+ (1-\psi_n) \left[(1+\sigma^2) \|y_n - x_n\|^2 + \left(1 + \frac{1}{\sigma^2}\right) \|x_n - x_{n+1}\|^2 \right] \\ \leq& \varphi^* \|y_n - x_n\|^2 + \xi_n, \end{aligned} \tag{4.7}$$

where $\varphi^* = (1-\psi_n)(1+\sigma) < 1$ and $\xi_n = (\delta_2 + \frac{1+2\sigma^2}{\sigma^2}) \|x_n - x_{n+1}\|^2 + \delta_2 (1 + \delta_2) \|x_n - x_{n-1}\|^2$. Since $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty$, then $\sum_{n=1}^{\infty} \xi_n < \infty$. Therefore, applying Lemma 2.4 in (4.7), we obtain

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{4.8}$$

Combining (4.4) and (4.8), we get that

$$\|y_{n+1} - x_n\| \le \|y_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{4.9}$$

Combining (4.5) and (4.8), we have

$$\|y_n - w_n\| \le \|y_n - x_n\| + \|x_n - w_n\| \to 0. \tag{4.10}$$

It follows from (4.5) and (4.9), we obtain

$$\|y_{n+1} - w_n\| \le \|y_{n+1} - x_n\| + \|x_n - w_n\| \to 0. \tag{4.11}$$

By the definition of y_{n+1} and (4.11), we deduce that

$$\|Ty_n - w_n\| \le \frac{1}{1-\psi_n} \|y_{n+1} - w_n\| \to 0. \tag{4.12}$$

Therefore, combining (4.5), (4.8) and (4.12), we obtain

$$\begin{aligned} \|Tx_n - x_n\| &\le \|Tx_n - Ty_n\| + \|Ty_n - w_n\| + \|w_n - x_n\| \\ &\le \|y_n - x_n\| + \|Ty_n - w_n\| + \|w_n - x_n\| \to 0. \end{aligned} \tag{4.13}$$

By (4.13) and Lemma 2.1, it follows that every weak limit point of $\{x_n\}$ is a fixed point of T, i.e., $\omega_w \{x_n\} \subset \mathrm{Fix}(T)$. This fact, with the inequality (4.3) and Lemma 2.3, ensures the strong convergence of $\{x_n\}$ to $P_{\mathrm{Fix}(T)} x_0$. This completes the proof. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$. *Let*

$$\delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset \left(\frac{\sigma}{1+\sigma}, 1\right), \sigma \in (0, 1). \tag{4.14}$$

Set $x_{-1}, x_0, y_0 \in C$ arbitrarily. Define two sequences $\{x_n\}$ and $\{y_n\}$ by the following algorithm:

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ y_{n+1} = \psi_n w_n + (1 - \psi_n) Ty_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_{n+1} - z\|^2 \le \psi_n \|w_n - z\|^2 + (1-\psi_n) \|y_n - z\|^2 \\ \qquad\qquad -\psi_n (1 - \psi_n) \|w_n - Ty_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, n \ge 0. \end{cases} \tag{4.15}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (4.15) *converge strongly to* $P_{\mathrm{Fix}(T)} x_0$.

Proof. This result can be easily proved by using a similar way as Theorem 4.1. We leave the proof for the reader to verify. □

Remark 4.3. (i) It should be pointed out that Algorithm (4.2) and Algorithm (4.15) are different from Algorithm (3.2) and Algorithm (3.17). In fact, the idea of Algorithm (4.2) and Algorithm (4.15) came from Malitsky and Semenov [19].

(ii) When $\delta_n = 0$, the Algorithm (4.2) is transformed into a new hybrid projection algorithm introduced by Dong et al. [13]. It should be noted that our conditions (4.1) are weaker than the conditions of Dong et al. [13].

Changing the definitions of y_{n+1} and C_n in Algorithm (4.2) and Algorithm (4.15), respectively, we get the following Theorem 4.4 and Theorem 4.5.

Theorem 4.4. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let*

$$\delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset \left(\frac{\sigma}{1+\sigma}, 1\right), \sigma \in (0,1).$$

Set $x_{-1}, x_0, y_0 \in C$ arbitrarily. Define two sequences $\{x_n\}$ and $\{y_n\}$ by the following algorithm:

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ y_{n+1} = \psi_n y_n + (1-\psi_n) T w_n, \\ C_n = \Big\{ z \in C : \|y_{n+1} - z\|^2 \leq \psi_n \|y_n - z\|^2 + (1-\psi_n) \|w_n - z\|^2 \\ \qquad\qquad -\psi_n (1-\psi_n) \|y_n - T w_n\|^2 \Big\}, \\ Q_n = \{ z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0 \}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geq 0. \end{cases} \tag{4.16}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (4.16) *converge strongly to $P_{\mathrm{Fix}(T)} x_0$.*

Theorem 4.5. *Let C be a nonempty closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive mapping with $\mathrm{Fix}(T) \neq \emptyset$. Let*

$$\delta_n \subset [\delta_1, \delta_2], \delta_1 \in (-\infty, 0], \delta_2 \in [0, \infty), \psi_n \subset \left(\frac{\sigma}{1+\sigma}, 1\right), \sigma \in (0,1).$$

Set $x_{-1}, x_0, y_0 \in C$ arbitrarily. Define two sequences $\{x_n\}$ and $\{y_n\}$ by the following algorithm:

$$\begin{cases} w_n = x_n + \delta_n (x_n - x_{n-1}), \\ y_{n+1} = \psi_n y_n + (1-\psi_n) T w_n, \\ C_{n+1} = \Big\{ z \in C_n : \|y_{n+1} - z\|^2 \leq \psi_n \|y_n - z\|^2 + (1-\psi_n) \|w_n - z\|^2 \\ \qquad\qquad -\psi_n (1-\psi_n) \|y_n - T w_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}} x_0, n \geq 0. \end{cases} \tag{4.17}$$

Then the sequences $\{x_n\}$ and $\{y_n\}$ defined by (4.17) *converge strongly to $P_{\mathrm{Fix}(T)} x_0$.*

Remark 4.6. It is important to highlight that, in expression form, Algorithm (4.16) and Algorithm (4.17) are different from Algorithm (3.2) and Algorithm (3.17) for nonexpansive mappings. However, through a simple calculation of C_n and C_{n+1} in Algorithm (4.16) and Algorithm (4.17), we can induce that Algorithm (3.2) and Algorithm (3.17) for nonexpansive mappings. That is to say, Algorithm (4.16) and Algorithm (4.17) are equivalent to Algorithm (3.2) and Algorithm (3.17) for nonexpansive mappings, respectively.

5. Numerical experiments

In this section, we provide two numerical examples to illustrate the computational performance of our proposed Algorithm (3.2), Algorithm (3.17), Algorithm (4.2) and Algorithm (4.15). All the programs are performed in MATLAB2018a on a PC Desktop Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.800 GHz, RAM 8.00 GB.

Example 5.1. For any nonempty closed convex set $C \subset R^N$, we consider the following variational inequality problem (in short, VI):

$$\text{find } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0, \quad \forall x \in C, \tag{5.1}$$

where $f : R^N \to R^N$ is a mapping. Denote by $\mathrm{VI}(C, f)$ the solution of the variational inequality (5.1). Define $T : R^N \to R^N$ by $T := P_C(I - \gamma f)$, where $0 < \gamma < 2/L$, L is the Lipschitz constant of the mapping f. Xu [32] showed that T is an averaged mapping, that is, T can be written as the average of the identity I and a nonexpansive mapping. It follows that $\mathrm{Fix}(T) = \mathrm{VI}(C, f)$. Therefore, we can solve VI (5.1) by finding the fixed point of T.

Taking $f : R^2 \to R^2$ as follows:

$$f(x, y) = (2x + 2y + \sin(x), -2x + 2y + \sin(y)), \quad \forall x, y \in R.$$

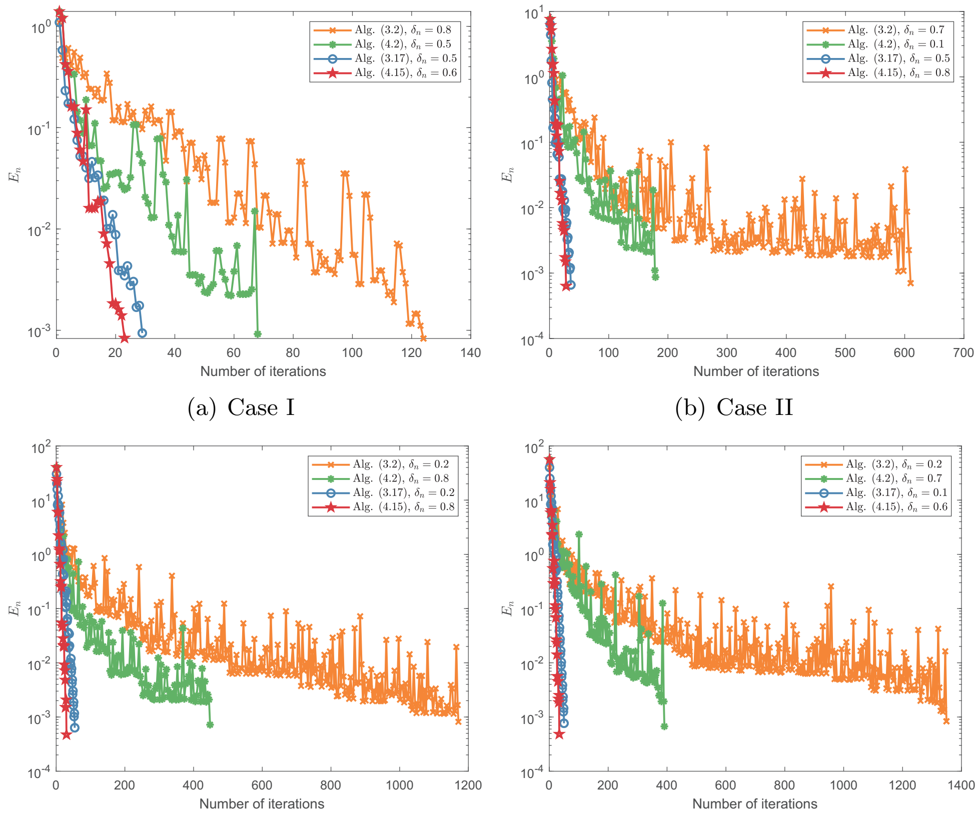
The feasible set C is given by $C = \{x \in R^2 | -10e \leq x \leq 10e\}$, where $e = (1, 1)^\mathsf{T}$. It is not hard to check that f is Lipschitz continuous with constant $L = \sqrt{26}$ and 1-strongly monotone. Therefore the VI (5.1) has a unique solution $x^* = (0, 0)^\mathsf{T}$.

Our parameters are set as follows. In all algorithms, set $\psi_n = 0.5$, $\gamma = 0.9/L$. Denote by $E_n = \|x_n - x^*\|_2$ the error of iterative algorithms and $E_n < 10^{-3}$ a common stopping criterion. Let $x_{-1} = x_0$, y_0 be randomly generated by the MATLAB function $k\times$rand(2,1) (where, Case I: $k = 5$, Case II: $k = 10$, Case III: $k = 50$, Case IV: $k = 100$). The numerical results are reported in Table 1 and Fig. 1. In Table 1, "Iter." denote the number of iterations. Table 1 shows the convergence behavior of iteration error E_n of our algorithms under different initial values and different inertial parameter. The convergence process of E_n with different initial values and inertial parameter are shown in Fig. 1.

Table 1. Computational results for Example 5.1

Algorithm	Case	δ_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Alg. (3.2)	Case I	Iter.	242	269	228	190	337	265	288	300	**124**	216
Alg. (4.2)			203	100	138	161	82	**68**	125	97	160	103
Alg. (3.17)			33	37	30	30	33	**29**	29	34	44	39
Alg. (4.15)			26	29	26	25	25	26	**23**	25	24	24
Alg. (3.2)	Case II	Iter.	1117	770	635	728	769	621	808	**610**	916	669
Alg. (4.2)			450	**179**	245	454	290	277	285	242	320	418
Alg. (3.17)			54	45	46	38	45	**36**	51	51	51	52
Alg. (4.15)			33	32	33	33	33	30	33	29	**28**	29
Alg. (3.2)	Case III	Iter.	1442	1917	**1171**	1851	1352	1333	1821	2118	1851	2224
Alg. (4.2)			669	829	1273	556	706	795	588	562	**448**	873
Alg. (3.17)			59	55	**54**	60	55	57	55	58	62	63
Alg. (4.15)			40	37	37	42	44	36	33	34	**30**	38
Alg. (3.2)	Case IV	Iter.	2728	1840	**1349**	1618	2322	1649	1447	1691	2380	4035
Alg. (4.2)			831	720	488	469	653	860	1056	**391**	664	1175
Alg. (3.17)			66	**50**	60	63	62	51	62	59	61	57
Alg. (4.15)			41	42	40	36	38	35	**34**	44	38	41

Example 5.2. Suppose that $H = L^2([0, 1])$ with inner product $\langle x, y \rangle := \int_0^1 x(t)y(t)\mathrm{d}t$ and norm $\|x\| := (\int_0^1 |x(t)|^2 \mathrm{d}t)^{1/2}$. Let $C := \{x \in H : \|x\| \leq 1\}$ be the unit ball.



(a) Case I (b) Case II

(c) Case III (d) Case IV

FIGURE 1. Convergence behavior of iteration error $\{E_n\}$ for Example 5.1

Define an operator $f : C \to H$ by

$$f(x)(t) = \int_0^1 (x(t) - G(t,v)g(x(v)))\, \mathrm{d}v + h(t), \quad t \in [0,1], x \in C,$$

where

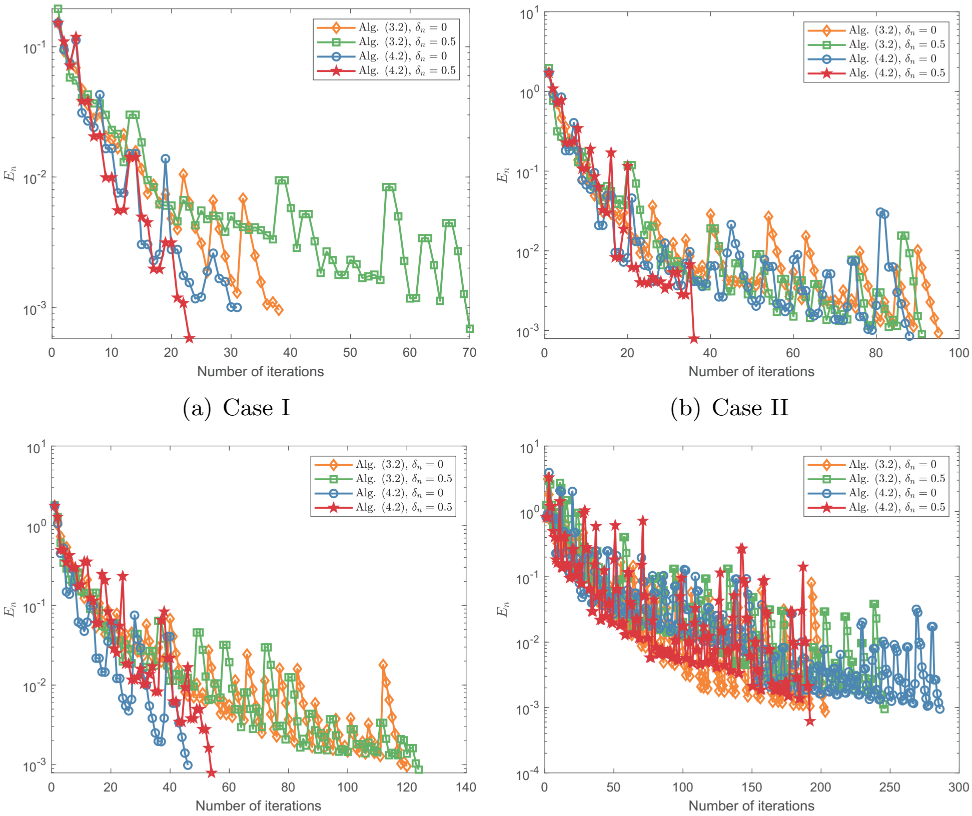
$$G(t,v) = \frac{2tv\mathrm{e}^{t+v}}{e\sqrt{\mathrm{e}^2 - 1}}, \quad g(x) = \cos x, \quad h(t) = \frac{2t\mathrm{e}^t}{e\sqrt{\mathrm{e}^2 - 1}}.$$

It is known that f is monotone and L-Lipschitz continuous with $L = 2$ and $x^* = \{\mathbf{0}\}$ is the solution of the corresponding variational inequality problem.

We use Algorithm (3.2) and Algorithm (4.2) to solve Example 5.2. Our parameters and stopping criteria are set the same as in Example 5.1. Numerical results are reported in Table 2 and Fig. 2. In Table 2, "Iter." and "Time(s)" denote the number of iterations and the CPU time in seconds, respectively.

TABLE 2. Computational results for Example 5.2

		Alg. (3.2), $\delta_n = 0$		Alg. (3.2), $\delta_n = 0.5$		Alg. (4.2), $\delta_n = 0$		Alg. (4.2), $\delta_n = 0.5$	
Cases	Initial values	Iter.	Time(s)	Iter.	Time(s)	Iter.	Time(s)	Iter.	Time(s)
I	$x_{-1} = \cos(t)$ $x_0 = y_0 = \sin(t)$	38	26.7584	70	49.6868	31	23.8894	23	17.2378
II	$x_{-1} = 5\,t^2$ $x_0 = y_0 = 5\,2^t/4$	95	68.2707	91	62.0445	88	64.5869	36	26.3965
III	$x_{-1} = 5\sin(t)$ $x_0 = y_0 = e^t$	120	80.4324	124	82.2207	46	32.5016	54	40.3045
IV	$x_{-1} = 10\,e^{t/2}$ $x_0 = y_0 = 5t^2/2$	203	135.1466	246	168.6629	286	206.6866	192	138.6253



(a) Case I (b) Case II

(c) Case III (d) Case IV

FIGURE 2. Convergence behavior of iteration error $\{E_n\}$ for Example 5.2

Remark 5.3. (i) In Example 5.1 and Example 5.2, we find that Algorithm (3.17) and Algorithm (4.15) have less oscillating behavior and enjoy faster convergence rates than Algorithm (3.2) and Algorithm (4.2), respectively. In addition, it should be noted that the choice of initial values will not affect the computational performance of our algorithms.

(ii) From Table 1 we get that inertial parameters have different effects on our algorithms. That is to say, algorithms with inertial term are not necessarily faster than those without inertial term. It should be mentioned that under the correct choice of inertial parameters, Algorithm (3.2), Algorithm (3.17), Algorithm (4.2) and Algorithm (4.15) have faster convergence rates than those without inertial term. See Table 1 and Fig. 1.

(iii) From Table 1 and Table 2, when $\delta_n = 0$, it should be pointed out that Algorithm (4.2) and Algorithm (4.15) have faster convergence rates than Algorithm (3.2) and Algorithm (3.17), respectively. In fact, as shown in Table 1 and Table 2, in most cases Algorithm (4.2) and Algorithm (4.15) are better than Algorithm (3.2) and Algorithm (3.17), respectively. It should be highlighted that research on the Algorithm (4.2) and the Algorithm (4.15) is very preliminary.

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