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RESEARCH ARTICLE

Modified inertial extragradient methods for finding minimum-norm solution of the variational inequality problem with applications to optimal control problem

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ABSTRACT

In order to discover the minimum-norm solution of the pseudomonotone variational inequality problem in a real Hilbert space, we provide two variants of the inertial extragradient approach with a novel generalized adaptive step size. Two of the suggested algorithms make use of the projection and contraction methods. We demonstrate several strong convergence findings without requiring the prior knowledge of the Lipschitz constant of the mapping. Finally, we give a number of numerical examples that highlight the benefits and effectiveness of the suggested algorithms and how they may be used to solve the optimal control problem.

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1. Introduction

The primary goal of this study is to construct several accelerated iterative methods with adaptive step sizes for finding the solutions of variational inequality problems in infinite-dimensional Hilbert spaces. Let $A : \mathcal{H} \to \mathcal{H}$ be an operator and let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Take $C \subset \mathcal{H}$ is a nonempty, closed, and convex subset of $\mathcal{H}$. The *variational inequality problem* (shortly, VIP) is find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C. \tag{VIP}$$

Variational inequality theory provides a fundamental model for many areas; for example engineering, economics, traffic management, operations optimization, and mathematical programming, and it constructs a unified framework for many optimization problems (see, e.g. [1,6,22,28,42]). Therefore, the theory and solution methods of variational inequalities have received more and more attention from scholars.

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A vast variety of numerical approaches for solving variational inequality problems have been presented throughout the last few decades. Next, we review some known methods in the literature for solving variational inequalities in finite- and infinite-dimensional spaces, which motivate us to propose new iterative algorithms. The Korpelevich extragradient method [15], which calls for computing the projection on the feasible set twice in each iteration, is the oldest and simplest method for dealing with the variational inequality problem. It is well known that computing projections may be challenging, particularly when the structure of the feasible set is intricate. Some approaches that only need computing the projection on the feasible set once per iteration have been developed to solve this problem; see, e.g. [3,11,40]. The main idea of these methods is to replace the iterative process of the second step in the extragradient method with a display calculation. Numerous variations based on these techniques [3,11,40] have recently been presented (see, e.g. [14,24,29,30,34,36,39,43]). Their numerical tests demonstrate the computational effectiveness and benefits of the suggested algorithms.

Recently, inspired by the work of Dong, Jiang and Gibali [8], Thong and Gibali [32] proposed the following Algorithm 1.1 to solve VIP in Hilbert spaces. On the other hand, Gibali, Thong and Tuan [10] also proposed the following Algorithm 1.2 for solving the monotone variational inequality problem based on the projection and contraction method [11].

Algorithm 1.1

Initialization: Given $\lambda > 0$, $l \in (0,1)$, $\mu \in (0,1)$, and $\gamma \in (0,2)$.
Iterative Steps: Let $x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Compute $v_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is chosen to be the largest $\kappa \in \{\lambda, \lambda l, \lambda l^2, \dots\}$ satisfying

$$\kappa \|Ax_n - Av_n\| \leq \mu \|x_n - v_n\| \tag{1}$$

If $x_n = v_n$ then stop and v_n is a solution of (VIP). Otherwise, go to **Step 2**.
Step 2. Compute $z_n = P_{T_n}(x_n - \gamma \lambda_n \rho_n A v_n)$, where $T_n := \{x \in \mathcal{H} : \langle x_n - \lambda_n A x_n - v_n, x - v_n \rangle \leq 0\}$, and

$$\rho_n := (1-\mu)\frac{\|x_n - v_n\|^2}{\|g_n\|^2}, \quad g_n := x_n - v_n - \lambda_n (Ax_n - Av_n). \tag{2}$$

Step 3. Compute $x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n z_n$.
Set $n := n + 1$ go to **Step 1**.

Algorithm 1.2

Initialization: Given $\lambda > 0$, $l \in (0,1)$, $\mu \in (0,1)$, and $\gamma \in (0,2)$.
Iterative Steps: Let $x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Compute $v_n = P_C(x_n - \lambda_n A x_n)$, where λ_n is generated by (1).
Step 2. Compute $z_n = x_n - \gamma \rho_n g_n$, where ρ_n and g_n are defined in (2).
Step 3. Compute $x_{n+1} = (1 - \alpha_n - \beta_n)x_n + \beta_n z_n$.
Set $n := n + 1$ go to **Step 1**.

The strong convergence theorems for the suggested iterative techniques in infinite-dimensional Hilbert spaces were obtained by Thong and Gibali [32] and Gibali et al. [10], respectively, under some reasonable restrictions imposed on the mapping and parameters. It is important to keep in mind that the Algorithms 1.1 and 1.2 only need to perform the projection on the feasible set once throughout each iteration. Their numerical tests demonstrate that the suggested algorithms outperform the existing approaches [3,8,24] in terms of computational efficiency and accuracy. Furthermore, we note that the Algorithms 1.1 and 1.2 employ an Armijo-type line search step size criterion enabling them to operate without requiring prior knowledge of the Lipschitz constant of the mapping. However, using Armijo-type step sizes may require the proposed algorithm to calculate the projection values on the feasible set multiple times per iteration. To overcome this drawback, Yang and Liu [46] introduced a new adaptive step size criterion which only needs to use some previously known information to complete the calculation of the step size. Recently, many scholars have used the idea of this criterion to construct numerous algorithms for finding the solutions of variational inequalities and equilibrium problems; see, e.g. [9,16,31,33,36,45,47].

Many scholars have focussed a lot of their attention and study on the concept of inertial as one of the ways of acceleration. The primary characteristic of inertial-type approaches is that the combination of the previous two (or more) iterations determines the outcome of the subsequent iteration. It has been observed that this minor adjustment might accelerate the convergence of inertial-free algorithms. Numerous inertial-type methods have been developed to handle variational inequalities, equilibrium problems, split feasibility problems, fixed point problems, inclusion problems, and others (see, e.g. [4,7,12,23,25,26,29,31,35,36,43]). Numerous numerical simulations show the benefits and effectiveness of their inertial methods compared to the version without inertial terms.

In this paper, we suggest two adaptive algorithms with inertial terms to handle variational inequality problems in real Hilbert spaces, inspired and motivated by the aforementioned findings. We made the following contributions to this research.

- Our two algorithms use a new step size without any line search procedure, which generalizes the step size suggested by Liu and Yang [16]. In addition, our two adaptive algorithms are preferable to the fixed-step algorithms suggested in [4,35]. Numerical experimental results show that our step size is useful and efficient, and that our two algorithms require less execution time than the algorithms in [10,32] that use the Armijo step size.
- Our two algorithms are designed to solve pseudo-monotone variational inequality problems, which improves the results used in [8,10,24,32,45,46] for finding the solutions of monotone variational inequalities.
- To accelerate the convergence speed of the proposed algorithms, the inertial term is also embedded in our algorithms. Numerical experimental results demonstrate that the proposed algorithms converge faster than the methods without inertial in [10,32].
- The strong convergence theorems of the proposed algorithms are proved under some suitable conditions. This improves the weak convergence results obtained in [3,8,16,25].
- To demonstrate the benefits and computational effectiveness of the suggested methods in comparison to those that were previously known in [10,32], several numerical experiments and applications in optimal control problems are provided.

The rest of this paper is structured as follows. Basic definitions and lemmas that should be utilized are gathered in Section 2. In Section 3, we describe two new non-monotonic inertial extragradient algorithms and examine their convergence. In Section 4, a few numerical tests are provided to demonstrate the benefits and effectiveness of the suggested algorithms. In Section 5, we solve the optimal control problem utilizing the suggested methods. Finally, Section 6 provides a succinct review of the research.

2. Preliminaries

The following equality and inequality are useful for our proofs.

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \quad \forall\, x, y \in \mathcal{H}, \tag{3}$$

and

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle, \quad \forall\, x, y \in \mathcal{H}. \tag{4}$$

Let $C \subset \mathcal{H}$ be a nonempty, closed, and convex. Recall that the *metric projection* of $\mathcal{H}$ onto C, denoted by P_C, which is defined as for any $x \in \mathcal{H}$, there exists a unique nearest point in C, given as $P_C(x)$ such that

$$\|x - P_C(x)\| \le \|x-y\|, \quad \forall\, y \in C.$$

Note that P_C has following properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0, \quad \forall\, x \in \mathcal{H},\ y \in C, \tag{5}$$

and

$$\|x-y\|^2 \ge \|x - P_C(x)\|^2 + \|y - P_C(x)\|^2, \quad \forall\, x \in \mathcal{H},\ y \in C. \tag{6}$$

Let $\mathrm{VIP}(C, A)$ denote the solution set of the variational inequality problem (VIP). It is easy to check the following relation according to (5).

$$z \in \mathrm{VI}(C, A) \ \Leftrightarrow\ z = P_C(z - \lambda Az), \quad \forall\, \lambda > 0. \tag{7}$$

Definition 2.1: A mapping $A : C \to \mathcal{H}$ is said to be:

(1) *monotone* if $\langle Ax - Ay, x - y\rangle \ge 0$ for all $x, y \in C$;
(2) *pseudomonotone* if $\langle Ax, y - x\rangle \ge 0$, we have $\langle Ay, y - x\rangle \ge 0$ for all $x, y \in C$;
(3) *L-Lipschitz continuous* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \le L\|x - y\|$ for all $x, y \in C$;
(4) *sequentially weakly continuous* on C if, for each sequence $\{x_n\} \subset C$ such that $x_n \rightharpoonup x$, we have $Ax_n \rightharpoonup Ax$.

Remark 2.1: From the above definitions, we see that (1) $\Rightarrow$ (2), but the converse is not true in general (see, e.g. [27, Example 4.2]).

Lemma 2.1 ([5]): *Let $C \subset \mathcal{H}$ be a nonempty closed and convex set and $A : C \to \mathcal{H}$ be a pseudomonotone and continuous mapping. Then z is a solution of the problem* (VIP) *if and only if*

$$\langle Ax, x - z\rangle \ge 0, \quad \forall\, x \in C.$$

Lemma 2.2 ([17]): *Let $\{a_n\}$ and $\{c_n\}$ be sequences of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \delta_n)a_n + b_n + c_n, \quad \forall\, n \ge 1,$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence. Assume that $\sum_{n=0}^{\infty} c_n < \infty$. Then the following results hold:

(1) *If $b_n \le \delta_n M$ for some $M \ge 0$, then $\{a_n\}$ is a bounded sequence.*
(2) *If $\sum_{n=0}^{\infty} \delta_n = \infty$ and $\limsup_{n\to\infty} \frac{b_n}{\delta_n} \le 0$, then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.3 ([18]): *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for each $i \in \mathbb{N}$. Define the sequence $\{\kappa(n)\}_{n \ge n_0}$ of integers as follows:*

$$\kappa(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ such that $\{k \le n_0 : \Gamma_k < \Gamma_{k+1}\} \ne \emptyset$. Then the following results hold:

(1) $\kappa(n_0) \leq \kappa(n_0+1) \leq \cdots$ *and* $\kappa(n) \to \infty$.
(2) $\Gamma_{\kappa(n)} \leq \Gamma_{\kappa(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\kappa(n)+1}$ *for each* $n \geq n_0$.

3. Main results

We make the following assumptions about our algorithms in order to prove some strong convergence theorems for them:

(A1) The feasible set C is a closed and convex subset of a real Hilbert space $\mathcal{H}$;
(A2) The mapping $A : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous and pseudomonotone on $\mathcal{H}$;
(A3) The mapping $A : \mathcal{H} \to \mathcal{H}$ satisfies the following condition: for each $\{t_n\} \subset C$ such that $t_n \rightharpoonup x$,

$$\|Ax\| \leq \liminf_{n\to\infty} \|At_n\|; \tag{8}$$

(A4) The solution set of the problem (VIP) is nonempty, that is, $\Omega := \mathrm{VIP}(C,A) \neq \emptyset$, where $\mathrm{VIP}(C,A)$ denotes the solution set of the problem (VIP);
(A5) The positive sequence $\{\xi_n\}$ satisfies $\lim_{n\to\infty} \frac{\xi_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Remark 3.1: (1) For Assumption (A2), it suffices to assume that the mapping A is continuous pseudomonotone if $\mathcal{H}$ is a finite-dimensional Hilbert space and it is not necessary to assume A satisfies (8).
(2) Note that Assumption (A3) is weaker than the sequential weak continuity of the mapping A, which often assumed in many recent works related to the pseudomonotone problem (VIP) (see, for example, [4,14,29,34,36,39,43]). Indeed, let $A : \mathcal{H} \to \mathcal{H}$ be a mapping define by $Ax = x\|x\|$ for all $x \in \mathcal{H}$. It can be shown that A satisfies Assumption (A3), but not sequentially weakly continuous (see [21,38]). However, if A is monotone, then Assumption (A3) can be removed.

Now, we are in a position to describe the proposed Algorithm 3.1.
The following lemma is crucial for proving the convergence results.

Lemma 3.1: *Let* $\{\lambda_n\}$ *be a sequence generated by* (11). *Then there exists* $\lambda \in [\min\{\frac{\mu}{L}, \lambda_0\}, \lambda_0 + \sum_{n=1}^{\infty} p_n]$ *such that* $\lambda = \lim_{n\to\infty} \lambda_n$.

Proof: The proof of this lemma follows as that of Lemma 3.1 in [44], so we omit it here. ■

Remark 3.2: The adaptive step size in this work is different from the studied adaptive step size as in many works. In particular, if $p_n = 0$ and $q_n = 1$ for all $n \geq 0$, then the step size reduces to the step size of many methods (see, e.g. [9,33,36,45–47]). In addition, if $p_n \neq 0$ and $q_n = 1$ for all $n \geq 0$, then the step size becomes the step size in [16].

Lemma 3.2: *Let* $\{r_n\}$, $\{v_n\}$ *and* $\{g_n\}$ *be the sequences generated by Algorithm* 3.1. *If* $r_n = v_n$ *or* $g_n = 0$, *then* $v_n \in \Omega$.

Proof: By the definition of g_n, we have

$$\begin{aligned}\|g_n\| &= \|r_n - v_n - \lambda_n(Ar_n - Av_n)\| \\ &\geq \|r_n - v_n\| - \lambda_n\|Ar_n - Av_n\| \\ &\geq \|r_n - v_n\| - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\|r_n - v_n\| \\ &= \left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|.\end{aligned}$$

Algorithm 3.1 Modified inertial subgradient extragradient method

Initialization: Given $\lambda_0 > 0$, $\phi > 0$, $\sigma > 1$, $\gamma \in (0, \frac{2}{\sigma})$ and $\mu \in (0,1)$. Choose $\{p_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} p_n < \infty$ and $\{q_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} q_n = 1$.
Iterative Steps: Let $x_{-1}, x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 0)$. Set

$$r_n = (1-\alpha_n)(x_n + \phi_n(x_n - x_{n-1})),$$

where

$$\phi_n = \begin{cases} \min\left\{\dfrac{\xi_n}{\|x_n - x_{n-1}\|}, \phi\right\}, & \text{if } x_n \neq x_{n-1}, \\ \phi, & \text{otherwise.} \end{cases} \tag{9}$$

Step 2. Compute

$$v_n = P_C(r_n - \lambda_n A r_n).$$

If $r_n = v_n$ or $Av_n = 0$, then stop and v_n is a solution of the problem (VIP). Otherwise, go to **Step 3**.
Step 3. Compute

$$x_{n+1} = P_{T_n}(r_n - \gamma\lambda_n\rho_n A v_n),$$

where $T_n := \{x \in \mathcal{H} : \langle r_n - \lambda_n A r_n - v_n, x - v_n\rangle \leq 0\}$ and ρ_n is defined as follows:

$$\rho_n := (1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|^2}, \quad g_n := r_n - v_n - \lambda_n(Ar_n - Av_n), \tag{10}$$

and update the step size by

$$\lambda_{n+1} = \min\left\{\lambda_n + p_n, \frac{q_n\mu\|r_n - v_n\|}{\|Ar_n - Av_n\|}\right\}. \tag{11}$$

Set $n := n+1$ go to **Step 1**.

We can also show that

$$\|g_n\| \leq \left(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|.$$

Therefore, we conclude that

$$\left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\| \leq \|g_n\| \leq \left(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|. \tag{12}$$

By Lemma 3.1, one sees that $\lim_{n\to\infty}\lambda_n$ exists, which together with $\lim_{n\to\infty} q_n = 1$ gives

$$\lim_{n\to\infty}\frac{q_n\lambda_n}{\lambda_{n+1}} = 1.$$

Therefore, there exists a constant n_0 such that $1 - \frac{q_n\mu\lambda_n}{\lambda_{n+1}} > 0$ for all $n \geq n_0$. Hence we have that $r_n = v_n$ if and only if $g_n = 0$ by means of (12). If $r_n = v_n$, then $v_n = P_C(v_n - \lambda_n A v_n)$. This means that $v_n \in \Omega$ by means of (5). ∎

Lemma 3.3: *Suppose that Assumptions* (A1)–(A4) *hold. Let* $\{x_n\}$ *be formed by Algorithm* 3.1. *Then, for each* $p \in \Omega$ *and* $n \geq n_0$, *we have*

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2,$$

where $\chi_n := (\frac{1-\mu}{1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}}})^2$.

Proof: Let $p \in \Omega$. Then it follows from (6) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|r_n - \gamma\lambda_n\rho_n Av_n - p\|^2 - \|r_n - \gamma\lambda_n\rho_n Av_n - x_{n+1}\|^2 \\
&= \|r_n - p\|^2 - 2\gamma\lambda_n\rho_n\langle r_n - p, Av_n\rangle + \gamma^2\lambda_n^2\rho_n^2\|Av_n\|^2 - \|r_n - x_{n+1}\|^2 \\
&\quad + 2\gamma\lambda_n\rho_n\langle r_n - x_{n+1}, Av_n\rangle - \gamma^2\lambda_n^2\rho_n^2\|Av_n\|^2 \\
&= \|r_n - p\|^2 - \|r_n - x_{n+1}\|^2 - 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - p\rangle \\
&= \|r_n - p\|^2 - \|r_n - x_{n+1}\|^2 - 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle - 2\gamma\lambda_n\rho_n\langle Av_n, v_n - p\rangle.
\end{aligned}$$

Since $p \in \Omega$ and $v_n \in C$, one has $\langle Ap, v_n - p\rangle \geq 0$. Then, by the pseudomonotonicity of A, we have $\langle Av_n, v_n - p\rangle \geq 0$. Hence we have

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \|r_n - x_{n+1}\|^2 - 2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle. \tag{13}$$

It is clear that $x_{n+1} \in T_n$ and hence

$$\begin{aligned}
&-2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle \\
&= 2\gamma\rho_n\underbrace{\langle r_n - \lambda_n Ar_n - v_n, x_{n+1} - v_n\rangle}_{\leq 0} - 2\gamma\rho_n\langle r_n - v_n - \lambda_n(Ar_n - Av_n), x_{n+1} - v_n\rangle \\
&\leq -2\gamma\rho_n\langle r_n - v_n - \lambda_n(Ar_n - Av_n), x_{n+1} - v_n\rangle \\
&= -2\gamma\rho_n\langle g_n, x_{n+1} - v_n\rangle \\
&= -2\gamma\rho_n\langle g_n, r_n - v_n\rangle + 2\gamma\rho_n\langle g_n, r_n - x_{n+1}\rangle.
\end{aligned} \tag{14}$$

Now, we estimate $-2\gamma\rho_n\langle g_n, r_n - v_n\rangle$ and $2\gamma\rho_n\langle g_n, r_n - x_{n+1}\rangle$. By the definition of g_n and (11), we have

$$\begin{aligned}
\langle g_n, r_n - v_n\rangle &= \langle r_n - v_n - \lambda_n(Ar_n - Av_n), r_n - v_n\rangle \\
&\geq \|r_n - v_n\|^2 - \lambda_n\|(Ar_n - Av_n)\|\|r_n - v_n\| \\
&\geq \left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|^2.
\end{aligned}$$

Since $\lim_{n\to\infty}(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}) = 1 - \mu > \frac{1-\mu}{\sigma} > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0, \quad \forall\, n \geq n_0.$$

Thus we deduce

$$\langle g_n, r_n - v_n\rangle \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|^2, \quad \forall\, n \geq n_0.$$

Since $\rho_n = (1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|^2}$, we have $\rho_n \|g_n\|^2 = (1-\mu)\|r_n - v_n\|^2$. Therefore we obtain

$$-2\gamma\rho_n\langle g_n, r_n - v_n\rangle \le \frac{-2\gamma\rho_n^2}{\sigma}\|g_n\|^2, \quad \forall\, n \ge n_0. \tag{15}$$

On the other hand, it follows from the equality $2\langle a, b\rangle = \|a\|^2 + \|b\|^2 - \|a-b\|^2$ that

$$2\gamma\rho_n\langle g_n, r_n - x_{n+1}\rangle = \|r_n - x_{n+1}\|^2 + \gamma^2\rho_n^2\|g_n\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2. \tag{16}$$

Substituting (15) and (16) into (14), we obtain

$$-2\gamma\lambda_n\rho_n\langle Av_n, x_{n+1} - v_n\rangle \le \|r_n - x_{n+1}\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\rho_n^2\|g_n\|^2. \tag{17}$$

By the definition of g_n, we see that

$$\begin{aligned} \|g_n\| &\le \|r_n - v_n\| + \lambda_n\|Ar_n - Av_n\| \\ &\le \|r_n - v_n\| + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\|r_n - v_n\| \\ &= \left(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|. \end{aligned}$$

This implies that

$$\frac{1}{\|g_n\|^2} \ge \frac{1}{(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|r_n - v_n\|^2}.$$

Hence we have

$$\rho_n^2\|g_n\|^2 = (1-\mu)^2\frac{\|r_n - v_n\|^4}{\|g_n\|^2} \ge \frac{(1-\mu)^2}{(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2}\|r_n - v_n\|^2. \tag{18}$$

Combining (13), (17), and (18), we obtain

$$\|x_{n+1} - p\|^2 \le \|r_n - p\|^2 - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2, \quad \forall\, n \ge n_0, \tag{19}$$

where $\chi_n := \left(\frac{1-\mu}{1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}}}\right)^2$. ■

Lemma 3.4 ([37]): *Suppose that Assumptions (A1)–(A4) hold. Let $\{r_n\}$ be generated by Algorithm 3.1. If there exists a subsequence $\{r_{n_k}\} \subset \{r_n\}$ such that $\{r_{n_k}\}$ converges weakly to $v \in \mathcal{H}$ and $\lim_{k\to\infty}\|r_{n_k} - v_{n_k}\| = 0$, then $v \in \Omega$.*

Now, we prove the strong convergence of Algorithm 3.1.

Theorem 3.1: *Suppose that Assumptions (A1)–(A5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges strongly to $x^* = P_\Omega(0)$, where $\|x^*\| = \min\{\|x\| : x \in \Omega\}$.*

Proof: First, we show that $\{x_n\}$ is bounded. From Lemma 3.3 and $\gamma \in (0, \frac{2}{\sigma})$, one has

$$\begin{aligned}\|x_{n+1} - p\| &\leq \|r_n - p\| \\ &= \|(1-\alpha_n)(x_n - p + \phi_n(x_n - x_{n-1})) - \alpha_n p\| \\ &\leq (1-\alpha_n)\|x_n - p + \phi_n(x_n - x_{n-1})\| + \alpha_n\| - p\| \\ &\leq (1-\alpha_n)\|x_n - p\| + (1-\alpha_n)\phi_n\|x_n - x_{n-1}\| + \alpha_n\|p\|, \quad \forall\, n \geq n_0.\end{aligned}$$

Putting $\iota_n := (1-\alpha_n)\frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\| + \|p\|$ for all $n \geq n_0$. It is easy to see that $\lim_{n\to\infty} \iota_n$ exists, which implies that $\{\iota_n\}$ is bounded. Then by Lemma 2.2, one has $\{\|x_n - p\|\}$ is bounded. Note that

$$\|x_n\| \leq \|x_n - p + p\| \leq \|x_n - p\| + \|p\|.$$

Hence $\{x_n\}$ is bounded and consequently so are $\{r_n\}$ and $\{v_n\}$. Let x^* be the minimum-norm solution of Ω, that is, $x^* = P_\Omega(0)$. From (4), we have

$$\begin{aligned}\|r_n - x^*\|^2 &= \|(1-\alpha_n)(x_n - x^* + \phi_n(x_n - x_{n-1})) - \alpha_n x^*\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - x^* + \phi_n(x_n - x_{n-1})\|^2 + 2\alpha_n\langle x^*, x^* - r_n\rangle \\ &\leq (1-\alpha_n)^2\left(\|x_n - x^*\|^2 + 2\phi_n\langle x_n - x_{n-1}, x_n - x^* + \phi_n(x_n - x_{n-1})\rangle\right) \\ &\quad + 2\alpha_n\langle x^*, x^* - r_n\rangle \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x^* - r_n\rangle, \qquad (20)\end{aligned}$$

where $K_1 := \sup_{n\geq 0}\{\|x_n - x^* + \phi_n(x_n - x_{n-1})\|\}$. Putting (20) into (19), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq{}& (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x^* - r_n\rangle \\ & - \|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2, \qquad (21)\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq{}& (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x_{n+1} - r_n\rangle \\ & + 2\alpha_n\langle x^*, x^* - x_{n+1}\rangle \qquad (22)\end{aligned}$$

for all $n \geq n_0$. From (21), we have

$$\|r_n - x_{n+1} - \gamma\rho_n g_n\|^2 + \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_n\|r_n - v_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n K_2 \quad (23)$$

for all $n \geq n_0$, where $K_2 := \sup_{n\geq n_0}\{(1-\alpha_n)\frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\|K_1, \|x^*\|\|r_n - x^*\|\}$.

Now, we prove the strong convergence of $\{\|x_n - x^*\|^2\}$ converges to zero by consider the following two cases.

Case 1. Suppose there exists $N \in \mathbb{N}$ such that $\{\|x_n - x^*\|^2\}$ is monotonically nonincreasing for $n \geq N$. Since $\{\|x_n - x^*\|^2\}$ is bounded, we have $\{\|x_n - x^*\|^2\}$ converges and hence

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0.$$

Since $\gamma \in (0, \frac{2}{\sigma})$ and $\lim_{n\to\infty} \chi_n > 0$, it follows from (23) that

$$\lim_{n\to\infty} \|r_n - x_{n+1} - \gamma\rho_n g_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|r_n - v_n\| = 0. \tag{24}$$

For all $n \geq n_0$, we note that $\|g_n\| \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|$, which gives $\frac{1}{\|g_n\|} \leq \frac{\sigma}{(1-\mu)\|r_n - v_n\|}$. Hence we have

$$\begin{aligned}
\|r_n - x_{n+1}\| &\leq \|r_n - x_{n+1} - \gamma\rho_n g_n\| + \gamma\rho_n\|g_n\| \\
&= \|r_n - x_{n+1} - \gamma\rho_n g_n\| + \gamma(1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|} \\
&\leq \|r_n - x_{n+1} - \gamma\rho_n g_n\| + \gamma\sigma\|r_n - v_n\|.
\end{aligned}$$

Then it follows from (24) that

$$\lim_{n\to\infty} \|r_n - x_{n+1}\| = 0. \tag{25}$$

Moreover, we see that

$$\begin{aligned}
\|x_n - r_n\| &= \|(1-\alpha_n)\phi_n(x_n - x_{n-1}) - \alpha_n x_n\| \\
&\leq (1-\alpha_n)\phi_n\|x_n - x_{n-1}\| + \alpha_n\|x_n\| \\
&= \alpha_n\left((1-\alpha_n)\frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x_n\|\right).
\end{aligned}$$

Thus we have

$$\lim_{n\to\infty} \|x_n - r_n\| = 0. \tag{26}$$

It follows from (25) and (26) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - r_n\| + \|r_n - x_n\| \to 0 \quad \text{as } n \to \infty. \tag{27}$$

Since $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to some point $v \in \mathcal{H}$ such that

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle = \lim_{k\to\infty}\langle x^*, x^* - x_{n_k}\rangle = \langle x^*, x^* - v\rangle.$$

From (26), we also get $\{r_{n_k}\}$ converges weakly to $v \in \mathcal{H}$, which together with Lemma 3.4 and (24) implies that $v \in \Omega := \text{VIP}(C, A)$. From (5), we obtain

$$\limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle = \langle x^*, x^* - v\rangle \leq 0. \tag{28}$$

Moreover, from (27) and (28), we also get

$$\limsup_{n\to\infty}\langle x^*, x^* - x_{n+1}\rangle = \limsup_{n\to\infty}\langle x^*, x^* - x_n\rangle \leq 0. \tag{29}$$

This together with (22) and Lemma 2.2 yields that $\lim_{n\to\infty} \|x_n - x^*\|^2 \to 0$, that is, $x_n \to x^*$.

Case 2. Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define an integer sequence $\kappa(n)$ by $\kappa(n) := \max\{k \le n : \Gamma_k < \Gamma_{k+1}\}$ for all $n \ge n_0$ (for some n_0 large enough). By Lemma 2.3, $\{\kappa(n)\}$ is a nondecreasing sequence such that $\kappa(n) \to \infty$ as $n \to \infty$ and $\Gamma_{\kappa(n)} \le \Gamma_{\kappa(n)+1}$ for all $n \ge n_0$. Put $\Gamma_n := \|x_n - x^*\|^2$ for all $n \in \mathbb{N}$. By (23), one has

$$\begin{aligned}\|r_{\kappa(n)} - x_{\kappa(n)+1} - \gamma\rho_{\kappa(n)}g_{\kappa(n)}\|^2 &+ \gamma\left(\frac{2}{\sigma} - \gamma\right)\chi_{\kappa(n)}\|r_{\kappa(n)} - v_{\kappa(n)}\|^2 \\ &\le \|x_{\kappa(n)} - x^*\|^2 - \|x_{\kappa(n)+1} - x^*\|^2 + 2\alpha_{\kappa(n)}K_2 \\ &\le 2\alpha_{\kappa(n)}K_2,\end{aligned}$$

where $K_2 > 0$. Following similar argument as in Case 1, one has

$$\lim_{n\to\infty}\|r_{\kappa(n)} - x_{\kappa(n)+1} - \gamma\rho_{\kappa(n)}g_{\kappa(n)}\| = 0 \quad \text{and} \quad \lim_{n\to\infty}\|r_{\kappa(n)} - v_{\kappa(n)}\| = 0.$$

Moreover, we have

$$\lim_{n\to\infty}\|x_{\kappa(n)+1} - r_{\kappa(n)}\| = 0 \tag{30}$$

and

$$\limsup_{n\to\infty}\langle x^*, x^* - x_{\kappa(n)+1}\rangle \le 0. \tag{31}$$

From (22) and $\Gamma_{\kappa(n)} \le \Gamma_{\kappa(n)+1}$, one gets

$$\begin{aligned}\|x_{\kappa(n)+1} - x^*\|^2 &\le (1-\alpha_{\kappa(n)})\|x_{\kappa(n)} - x^*\|^2 + 2(1-\alpha_{\kappa(n)})\phi_{\kappa(n)}\|x_{\kappa(n)} - x_{\kappa(n)-1}\|K_1 \\ &\quad + 2\alpha_{\kappa(n)}\langle x^*, x_{\kappa(n)+1} - r_n\rangle + 2\alpha_{\kappa(n)}\langle x^*, x^* - x_{\kappa(n)+1}\rangle \\ &\le (1-\alpha_{\kappa(n)})\|x_{\kappa(n)+1} - x^*\|^2 + 2(1-\alpha_{\kappa(n)})\phi_{\kappa(n)}\|x_{\kappa(n)} - x_{\kappa(n)-1}\|K_1 \\ &\quad + 2\alpha_{\kappa(n)}\langle x^*, x_{\kappa(n)+1} - r_{\kappa(n)}\rangle + 2\alpha_{\kappa(n)}\langle x^*, x^* - x_{\kappa(n)+1}\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|x_{\kappa(n)+1} - x^*\|^2 &\le 2(1-\alpha_{\kappa(n)})\frac{\phi_{\kappa(n)}}{\alpha_{\kappa(n)}}\|x_{\kappa(n)} - x_{\kappa(n)-1}\|K_1 \\ &\quad + 2\|x_{\kappa(n)+1} - r_{\kappa(n)}\|\|x^*\| + 2\langle x^*, x^* - x_{\kappa(n)+1}\rangle,\end{aligned}$$

where $K_1 > 0$. Combining (30) and (31), we obtain

$$\lim_{n\to\infty}\|x_{\kappa(n)+1} - x^*\|^2 = 0.$$

By Lemma 2.3, we have

$$\|x_n - x^*\|^2 \le \|x_{\kappa(n)+1} - x^*\|^2 \to 0 \text{ as } n \to \infty.$$

Hence $x_n \to x^*$. Therefore we can conclude that $\{x_n\}$ converges strongly to the minimum-norm solution of (VIP) from the above two cases. ■

Next, we introduce the second modification of inertial extragradient method (see Algorithm 3.2 below) for solving pseudomonotone VIPs. This method motivated by the projection and contraction method [11] with a generalized adaptive step size.

Lemma 3.5: *Suppose that Assumptions* (A1)–(A4) *hold. Let* $\{x_n\}$ *be created by Algorithm* 3.2. *We have*

Algorithm 3.2 Modified inertial projection and contraction method

Initialization: Given $\lambda_0 > 0$, $\phi > 0$, $\sigma > 1$, $\gamma \in (0, \frac{2}{\sigma})$ and $\mu \in (0,1)$. Choose $\{p_n\} \subset [0,\infty)$ such that $\sum_{n=0}^{\infty} p_n < \infty$ and $\{q_n\} \subset [1,\infty)$ such that $\lim_{n\to\infty} q_n = 1$.
Iterative Steps: Let $x_{-1}, x_0 \in \mathcal{H}$ be arbitrary and calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 0)$. Set

$$r_n = (1-\alpha_n)(x_n + \phi_n(x_n - x_{n-1})),$$

where ϕ_n is defined in (9).
Step 2. Compute

$$v_n = P_C(r_n - \lambda_n A r_n).$$

If $r_n = v_n$ or $Av_n = 0$, then stop and v_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute

$$x_{n+1} = r_n - \gamma \rho_n g_n,$$

where ρ_n and g_n are defined in (10), and update the step size by (11).
Set $n := n+1$ go to **Step 1**.

(1) $\|x_{n+1}-p\|^2 \leq \|r_n - p\|^2 - \frac{1}{\gamma}(\frac{2}{\sigma}-\gamma)\|x_{n+1}-r_n\|^2$ *for each* $n \geq n_0$ *and* $p \in \Omega$;

(2) $\|r_n - v_n\|^2 \leq \chi_n'\|x_{n+1}-r_n\|^2$, *where* $\chi_n' := \left(\frac{1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\right)^2$.

Proof: (1) Let $p \in \Omega$, one sees that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|r_n - \gamma\rho_n g_n - p\|^2 \\ &= \|r_n - p\|^2 - 2\gamma\rho_n\langle r_n - p, g_n\rangle + \gamma^2\rho_n^2\|g_n\|^2. \end{aligned} \tag{32}$$

From the definition of g_n, we see that

$$\begin{aligned}\langle r_n - p, g_n\rangle &= \|r_n - v_n\|^2 - \lambda_n\langle r_n - v_n, Ar_n - Av_n\rangle + \langle v_n - p, r_n - v_n - \lambda_n(Ar_n - Av_n)\rangle \\ &\geq \|r_n - v_n\|^2 - \lambda_n\|r_n - v_n\|\|Ar_n - Av_n\| + \langle v_n - p, r_n - v_n - \lambda_n(Ar_n - Av_n)\rangle \\ &\geq \left(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|r_n - v_n\|^2 + \langle v_n - p, r_n - v_n - \lambda_n(Ar_n - Av_n)\rangle.\end{aligned}$$

According to $\lim_{n\to\infty}(1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}}) = 1-\mu > \frac{1-\mu}{\sigma} > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - q_n\mu\frac{\lambda_n}{\lambda_{n+1}} > \frac{1-\mu}{\sigma} > 0, \quad \forall\, n \geq n_0.$$

Thus we have

$$\langle r_n - p, g_n\rangle \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|^2 + \langle r_n - v_n - \lambda_n(Ar_n - Av_n), v_n - p\rangle, \quad \forall\, n \geq n_0. \tag{33}$$

Since $v_n = P_C(r_n - \lambda_n Ar_n)$ and from (5), one has

$$\langle r_n - \lambda_n Ar_n - v_n, v_n - p\rangle \geq 0.$$

Moreover, using $\langle Ap, v_n - p\rangle \geq 0$ and the pseudomonotonicity of A, one gets

$$\langle Av_n, v_n - p\rangle \geq 0.$$

Hence

$$\langle r_n - v_n - \lambda_n(Ar_n - Av_n), v_n - p\rangle = \underbrace{\langle r_n - \lambda_n Ar_n - v_n, v_n - p\rangle}_{\geq 0} + \lambda_n \underbrace{\langle Av_n, v_n - p\rangle}_{\geq 0} \geq 0. \quad (34)$$

Combining (33) and (34), we obtain

$$\langle r_n - p, g_n\rangle \geq \frac{1-\mu}{\sigma}\|r_n - v_n\|^2, \quad \forall\, n \geq n_0.$$

It follows from the definition of ρ_n that

$$\langle r_n - p, g_n\rangle \geq \frac{1}{\sigma}\rho_n\|g_n\|^2, \quad \forall\, n \geq n_0. \quad (35)$$

By using (33) and (36), one has

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \gamma\left(\frac{2}{\sigma} - \gamma\right)\rho_n^2\|g_n\|^2, \quad \forall\, n \geq n_0.$$

Since $x_{n+1} = r_n - \gamma\rho_n g_n$, we have $\rho_n^2\|g_n\|^2 = \frac{1}{\gamma^2}\|x_{n+1} - r_n\|^2$. Hence we have

$$\|x_{n+1} - p\|^2 \leq \|r_n - p\|^2 - \frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - r_n\|^2, \quad \forall\, n \geq n_0. \quad (36)$$

(2) By the definition of ρ_n, we have

$$\begin{aligned}\|r_n - v_n\|^2 &= \frac{1}{1-\mu}\cdot\rho_n\|g_n\|^2 = \frac{1}{1-\mu}\cdot\frac{1}{\gamma^2\rho_n}(\gamma^2\rho_n^2\|g_n\|^2)\\ &= \frac{1}{1-\mu}\cdot\frac{1}{\gamma^2\rho_n}\|x_{n+1} - r_n\|^2.\end{aligned} \quad (37)$$

From $\|g_n\| \leq (1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})\|r_n - v_n\|$, we have $\frac{1}{\|g_n\|^2} \geq \frac{1}{(1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2\|r_n - v_n\|^2}$. So

$$\rho_n = (1-\mu)\frac{\|r_n - v_n\|^2}{\|g_n\|^2} \geq \frac{1-\mu}{(1 + q_n\mu\frac{\lambda_n}{\lambda_{n+1}})^2}. \quad (38)$$

Combining (37) and (38), one has

$$\|r_n - v_n\|^2 \leq \chi_n'\|x_{n+1} - r_n\|^2, \quad (39)$$

where $\chi_n' := \left(\frac{1+q_n\mu\frac{\lambda_n}{\lambda_{n+1}}}{\gamma(1-\mu)}\right)^2$. ■

Theorem 3.2: *Suppose that Assumptions* (A1)–(A5) *hold. Then the sequence* $\{x_n\}$ *created by Algorithm* 3.2 *converges strongly to* $x^* = P_\Omega(0)$, *where* $\|x^*\| = \min\{\|x\| : x \in \Omega\}$.

Proof: From Lemma 3.5 and $\gamma \in (0, \frac{2}{\sigma})$, by using the same argument as in Theorem 3.1, we have that $\{x_n\}$ is bounded. Moreover, we can show that

$$\|r_n - x^*\|^2 \leq (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 + 2\alpha_n\langle x^*, x^* - r_n\rangle, \tag{40}$$

where $x^* = P_\Omega(0)$ and $K_1 > 0$. Putting (40) into (36), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 \\ &\quad + 2\alpha_n\langle x^*, x^* - r_n\rangle - \frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - r_n\|^2 \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + 2(1-\alpha_n)\phi_n\|x_n - x_{n-1}\|K_1 \\ &\quad + 2\alpha_n\langle x^*, x_{n+1} - r_n\rangle + 2\alpha_n\langle x^*, x^* - x_{n+1}\rangle\end{aligned} \tag{41}$$

for all $n \geq n_0$. From (41), we have

$$\frac{1}{\gamma}\left(\frac{2}{\sigma} - \gamma\right)\|x_{n+1} - r_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n K_2, \quad \forall\, n \geq n_0, \tag{42}$$

where $K_2 > 0$. Finally, we prove the strong convergence of $\{x_n\}$ converges to $x^* = P_\Omega(0)$ by consider the two cases, which are the same as in Theorem 3.1. Thus it follows from (42) that $\lim_{n\to\infty}\|x_{n+1} - r_n\| = 0$. This together with (39) gives that $\lim_{n\to\infty}\|r_n - v_n\| = 0$. The rest of the proof can be easily proved by similar arguments to that of Theorem 3.1 and so we omit it. ■

4. Numerical experiments

The purpose of this part is to illustrate the benefits and computing effectiveness of the suggested algorithms in comparison to several strongly convergent schemes in the literature [10,32]. The numerical examples take place in both finite- and infinite-dimensional spaces. The programmes are all executed in MATLAB 2018a using a PC with an Intel(R) Core(TM) i5-8250U CPU running at 1.60 GHz and 8.00 GB of RAM.

Example 4.1: Let $A : \mathbb{R}^m \to \mathbb{R}^m$ be given as $Ax := Gx + g$, where $g \in \mathbb{R}^m$ and $G := BB^\mathsf{T} + S + E$, matrix $B \in \mathbb{R}^{m\times m}$, matrix $S \in \mathbb{R}^{m\times m}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are nonnegative (hence G is positive symmetric definite). The feasible set C is given by $C := \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5, i = 1, 2, \dots, m\}$. It is easy to see that A is monotone (hence it is pseudomonotone) L-Lipschitz continuous with $L = \|G\|$. In this example, all entries of B, E are produced randomly in $[0, 2]$ and S is produced randomly in $[-2, 2]$. Let $g = \mathbf{0}$. Then the solution set is $x^* = \{\mathbf{0}\}$.

We compare the proposed algorithms with the following.

- Algorithm 3.1 in Thong and Gibali [32] (shortly, TG Alg. 3.1).
- Algorithm 3.1 in Gibali et al. [10] (shortly, GTT Alg. 3.1).

The parameters of our algorithms and the compared ones are set as follows.

- Taking $\lambda_0 = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 1/(n+1)$, $p_n = 1/(n+1)^{1.1}$, $q_n = (n+1)/n$, $\phi = 0.4$ and $\xi_n = 100/(n+1)^2$ for our Algorithms 3.1 and 3.2.
- Choosing $\lambda = 0.5$, $l = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 1/(n+1)$ and $\beta_n = 0.5(1-\alpha_n)$ for TG Alg. 3.1 and GTT Alg. 3.1.

The starting values $x_0 = x_1$ are produced at random using $5rand(m, 1)$ in MATLAB, and the maximum number of iterations 200 serves as a common stopping condition for all methods. At the nth

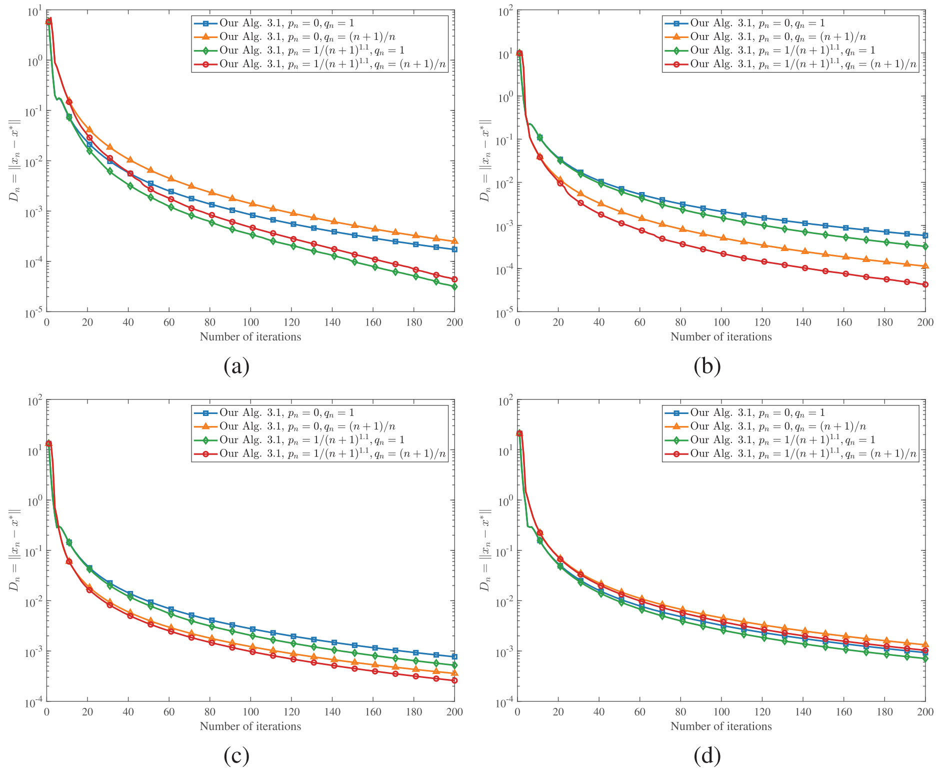

Figure 1. The behaviour of our Algorithm 3.1 for different p_n and q_n in Example 4.1. (a) $m = 20$. (b) $m = 50$. (c) $m = 100$ and (d) $m = 200$.

Table 1. Numerical results for all algorithms under different dimensions in Example 4.1.

	$m = 20$		$m = 50$		$m = 100$		$m = 200$	
Algorithms	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 3.1	2.09E−05	0.0349	4.42E−05	0.0273	3.74E−04	0.0337	1.09E−03	0.0419
Our Alg. 3.2	2.34E−05	0.0239	4.58E−05	0.0228	3.78E−04	0.0267	1.08E−03	0.0370
TG Alg. 3.1	1.11E−02	0.0430	3.49E−02	0.0412	5.77E−02	0.1538	8.88E−02	0.1683
GTT Alg. 3.1	1.11E−02	0.0370	3.49E−02	0.0364	5.77E−02	0.0709	8.88E−02	0.1286

step, we utilize $D_n := \|x_n - x^*\|$ to calculate the iteration error. First, we test the effect of different parameters p_n and q_n on the proposed algorithms with different dimensions, as shown in Figures 1 and 2. Next, Table 1 shows the results of the proposed methods compared to some known ones in different dimensions, where 'CPU' denotes the execution time in seconds.

Example 4.2: We consider an example in the Hilbert space $\mathcal{H} := L^2([0,1])$ associated with the inner product

$$\langle p, q \rangle := \int_0^1 p(t)q(t)\,\mathrm{d}t, \quad \forall\, p, q \in \mathcal{H},$$

and the induced norm

$$\|p\| := \left(\int_0^1 |p(t)|^2\,\mathrm{d}t \right)^{1/2}, \quad \forall\, p \in \mathcal{H}.$$

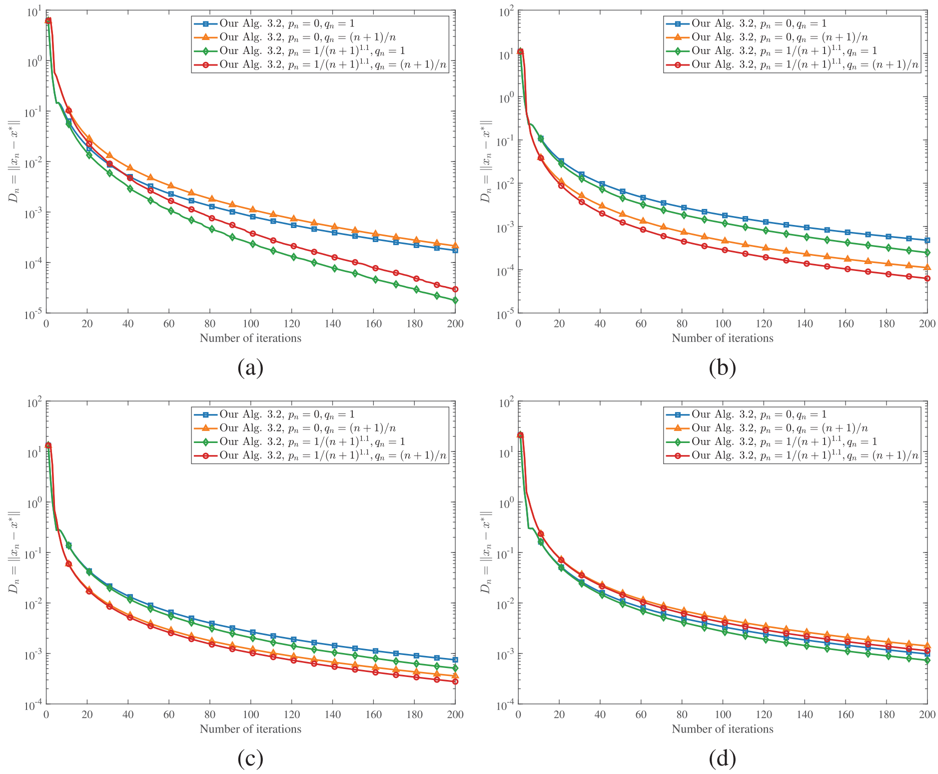

Figure 2. The behaviour of our Algorithm 3.2 for different p_n and q_n in Example 4.1. (a) $m = 20$. (b) $m = 50$. (c) $m = 100$ and (d) $m = 200$.

The feasible set is given by $C := \{x \in \mathcal{H} : \|x\| \leq 1\}$. Let $A : C \to \mathcal{H}$ be as follows.

$$(Ax)(t) := \int_0^1 \big(x(t) - Q(t,v)g(x(v))\big)\,\mathrm{d}v + h(t), \quad \forall\, t \in [0,1],\ x \in C,$$

where

$$Q(t,v) := \frac{2tv\,\mathrm{e}^{t+v}}{e\sqrt{\mathrm{e}^2 - 1}}, \quad g(x) := \cos x, \quad h(t) := \frac{2t\,\mathrm{e}^t}{e\sqrt{\mathrm{e}^2 - 1}}.$$

Note that A is monotone (hence it is pseudomonotone) and L-Lipschitz continuous with $L = 2$ (see [13] for more details) and $x^*(t) = \{\mathbf{0}\}$ is the solution of the (VIP).

The parameters of all algorithms are maintained the same as in Example 4.1. We utilize $D_n := \|x_n(t) - x^*(t)\|$ to calculate the iteration error of the nth step and set the maximum number of iterations for all algorithms to 50. The numerical behaviours of all algorithms with four starting points $x_0(t) = x_1(t)$ are reported in Table 2.

From Examples 4.1 and 4.2, we have the following observations.

(1) It can be seen from Figures 1 and 2 that the suggested methods have different impacts with different parameters p_n and q_n. Note that when $m = 50, 100$, the proposed algorithms on $q_n \neq 1$ has a higher accuracy than $q_n = 1$ when the values of p_n are the same. In addition, the proposed algorithms on $p_n \neq 0$ has a better performance than $p_n = 0$ when the values of q_n are the same. Thus, the iteration step sizes of the proposed algorithms are useful and efficient.

Table 2. Numerical results for all algorithms at different initial values in Example 4.2.

	$x_1 = 5t^3$		$x_1 = 4\sin(2t)$		$x_1 = 8\log(t)$		$x_1 = 3\exp(t)$	
Algorithms	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 3.1	8.44E−21	28.0391	8.80E−21	28.6204	1.83E−21	29.3688	3.27E−17	33.3884
Our Alg. 3.2	3.95E−21	26.4142	5.39E−22	27.1204	6.45E−18	27.3436	2.94E−13	34.7676
TG Alg. 3.1	7.47E−06	35.4475	1.02E−05	35.3399	2.68E−05	37.8135	1.50E−05	44.1810
GTT Alg. 3.1	6.70E−06	34.3776	8.30E−06	34.3631	2.05E−05	36.7857	1.25E−05	43.5128

(2) From Tables 1 and 2, we can obtain that our two algorithms have a better accuracy and less execution time than the algorithms presented in the literature [10,32]. These findings are independent of the size of the dimension and the choice of starting values. On the other hand, it is worth noting that the algorithms presented in [10,32] use an Armijo-type step size, which may lead them to require more execution time than our suggested adaptive algorithms.

5. Applications to optimal control problems

In this section, we use the proposed algorithms to solve the optimal control problem (see [20,29,41] for a description of this problem). Next, we run two tests in optimal control problems to illustrate the performance of our algorithms and compare them with the ones in [10,32]. The parameters of the algorithms are set as follows.

- Taking $\lambda_0 = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 10^{-4}/(n+1)$, $p_n = 10^{-1}/(n+1)^{1.1}$, $q_n = (n+1)/n$, $\phi = 0.01$ and $\xi_n = 10^{-4}/(n+1)^2$ for our Algorithms 3.1 and 3.2.
- Choosing $\lambda = 1$, $l = 0.5$, $\mu = 0.4$, $\gamma = 1.5$, $\alpha_n = 10^{-4}/(n+1)$ and $\beta_n = 0.5(1-\alpha_n)$ for TG Alg. 3.1 and GTT Alg. 3.1.

Example 5.1 (See [19]): Consider the following problem:

$$\begin{aligned} \text{minimize} \quad & x_2(3\pi) \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = -x_1(t) + u(t), \quad \forall\, t \in [0, 3\pi], \\ & x(0) = 0, \\ & u(t) \in [-1, 1]. \end{aligned}$$

The exact optimal control of Example 5.1 is known:

$$u^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2); \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

The initial controls $u_0(t) = u_1(t)$ are randomly generated in $[-1, 1]$ and the stopping criterion is either $D_n := \|u_{n+1} - u_n\| \le 10^{-4}$ or the maximum number of iterations is reached 1000. Figure 3 gives the approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.1.

We now consider an example in which the terminal function is not linear.

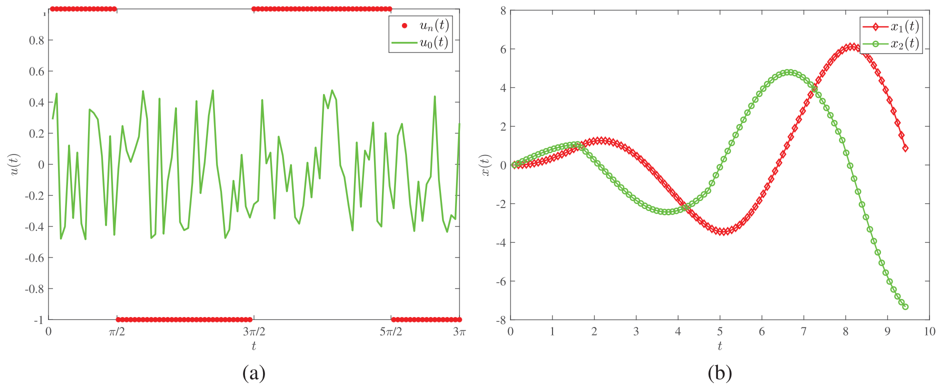

Figure 3. Numerical results of the proposed Algorithm 3.1 for Example 5.1. (a) Initial and optimal controls and (b) Optimal trajectories.

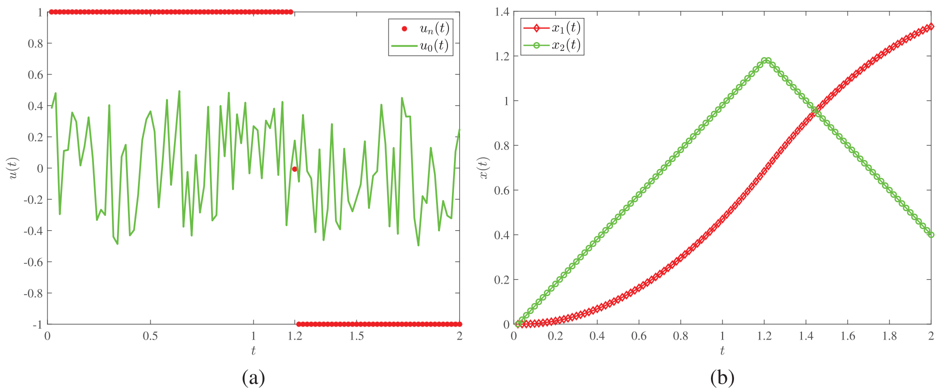

Figure 4. Numerical results of the proposed Algorithm 3.2 for Example 5.2. (a) Initial and optimal controls and (b) Optimal trajectories.

Example 5.2 (See [2]): Consider the following problem:

$$\begin{aligned}
\text{minimize} \quad & -x_1(2) + (x_2(2))^2, \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = u(t), \quad \forall\, t \in [0, 2], \\
& x_1(0) = 0, \quad x_2(0) = 0, \\
& u(t) \in [-1, 1].
\end{aligned}$$

The exact optimal control of Example 5.2 is known:

$$u^*(t) = \begin{cases} 1, & \text{if } t \in [0, 1.2); \\ -1, & \text{if } t \in (1.2, 2]. \end{cases}$$

The approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.2 are shown in Figure 4.

The results of our methods as well as the compared algorithms in Examples 5.1 and 5.2 are given in Table 3, where 'Iter.' represent the number of iterations.

Table 3. Numerical results for all algorithms in Examples 5.1 and 5.2.

	Example 5.1			Example 5.2		
Algorithms	Iter.	CPU	D_n	Iter.	CPU	D_n
Our Alg. 3.1	100	0.0468	9.9010E−05	175	0.0680	6.4170E−05
Our Alg. 3.2	111	0.0507	9.9305E−05	273	0.0823	8.7029E−05
TG Alg. 3.1	202	0.1245	9.9507E−05	417	0.1623	9.9175E−05
GTT Alg. 3.1	224	0.0856	9.9756E−05	1000	0.6143	2.4875E−04

From Figures 3, 4 and Table 3, it is clear that whether the terminal function is linear or nonlinear, the suggested techniques for solving optimal control problems can still produce satisfactory results. Additionally, compared to the algorithms described in the literature [10,32], they take fewer iterations and less time.

6. Conclusions

In this paper, two iterative approaches with a novel adaptive step size rule are suggested for locating the minimum-norm solution of a pseudomonotone variational inequality problem in a real Hilbert space. Without previous knowledge of the operator's Lipschitz constant, the strong convergence of the sequences produced by these methods has been demonstrated. To confirm the effectiveness and benefits of the suggested algorithms and to compare them with some related approaches in the literature, several numerical experiments have been carried out. Additionally, the optimum control problem has been investigated as an application of our main results.

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