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ADAPTIVE MODIFIED INERTIAL PROJECTION AND CONTRACTION METHODS FOR PSEUDOMONOTONE VARIATIONAL INEQUALITIES

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Abstract. To handle pseudomonotone variational inequality problems in real Hilbert spaces, four modified inertial projection and contraction algorithms with non-monotonic step sizes are suggested in this paper. The proposed algorithms take advantage of a novel non-monotonic step size criteria, allowing them to work without previous knowledge of the Lipschitz constant of the mapping involved. Under certain situations, the strong convergence of the iterative sequences generated by the suggested algorithms is established. Finally, several numerical experiments are offered to validate the theoretical conclusions.
Keywords. Inertial method; Pseudomonotone operator; Projection and contraction method; Subgradient extragradient method; Variational inequality.

1. INTRODUCTION

Let $\mathscr{X}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$ embedded with inner product $\langle\cdot\rangle$ and induced norm $\|\cdot\|$. Assume that $Q:\mathscr{X}\to\mathscr{H}$ is a nonlinear mapping. Recall that the classical variational inequality problem (shortly, VIP) for Q on $\mathscr{X}$ is presented below:

$$\text{find } b^*\in\mathscr{X} \text{ such that } \langle Qb^*, b-b^*\rangle \geq 0, \quad \forall b\in\mathscr{X}. \tag{VIP}$$

The solution set of (VIP) is denoted as Ω. In the past decades, the theory, models, and algorithms of variational inequality problems have received considerable attention and research interest from scholars due to their wide applications in engineering, operations research, economics, image reconstruction, optimal control, etc (see, e.g., [1, 2, 3]). On the other hand, it is known that a differentiable pseudoconvex function is characterized by the pseudomonotonicity of its gradient (see [4]). Thus the class of pseudomonotone operators is an extension of the class of differentiable pseudoconvex functions. In recent years, pseudomonotone operators attracted the interest of many scholars due to their applications in many important fields, such as fractional programming [5, 6], Nash equilibrium [7], and consumer theory of mathematical economics [8, 9]. The purpose of this paper is to investigate several efficient numerical methods to address the variational inequality problem with a pseudomonotone operator in real Hilbert spaces.

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Many iterative techniques for solving variational inequality problems in infinite-dimensional spaces have been proposed throughout the last few decades. Following that, we discuss some obstacles and limitations of various known algorithms for addressing variational inequality problems, which motivates us to propose some new ideas in this study. The projection-based technique is the focus of this paper. The projected gradient method (abbreviated PGM) is the most basic projection-type technique, as it calculates the projection on the feasible set only once per iteration. However, the convergence condition of the PGM demands that the operator involved is strongly monotonic, which may be difficult to meet in actual situations. To overcome this difficulty, Korpelevich [10] presented a two-step projection approach (now known as the extragradient method, abbreviated as the EGM) to solve the variational inequality problem. The convergence of the EGM is verified when the operator involved is monotone or pseudomonotone. It should be noted that the EGM needs the projection on the feasible set to be computed twice for each iteration. Because it is difficult to compute the projection onto the feasible set (especially when it is complicated), the EGM increases its computational burden compared to the PGM that only computes the projection once. A natural idea is whether it is possible to combine the computational efficiency of the PGM with the convergence advantage of the EGM. That is, *is there a method that computes the projection on the feasible set only once and can obtain convergence if the operator involved is monotone (or even pseudomonotone)*? The answer is affirmative. Recently, various numerical approaches for solving variational inequality problems that only need to calculate the projection on the feasible set once were presented; see, for example, the projection and contraction method [11], the Tseng's extragradient method [12], the subgradient extragradient method [13, 14, 15], and the projected gradient method [16]. In the situation that the operator is monotone, the weak convergence of these approaches was demonstrated.

Recently, many iterative methods based on the techniques considered in [11, 12, 13, 14, 15, 16] were presented to find solutions of variational inequalities; see, e.g., [17, 18, 19, 20, 21, 22, 23, 24, 25]. Note that the step sizes of the methods in [18, 21, 22] are related to the Lipschitz constant of the mapping involved, which indicates that the knowledge of this constant is required as a requirement for their work. However, this constant is not always available in practice, which would lead to the failure of the fixed-step algorithms in [18, 21, 22]. To address this limitation, two types of step size update criteria were proposed: one is an Armijo-type line search strategy (see, e.g., [17, 19, 20, 25]), and the other is an adaptive technique that makes use of previously known knowledge to perform a simple computation (see, e.g., [23, 24]). It should be emphasized that the Armijo-type criterion may need to compute the projection on the feasible set multiple times in each iteration to find a suitable step size. The disadvantage of the adaptive methods in [23, 24] is that they generate a non-increasing sequence of step sizes during the iteration. The use of these two types of step size criteria may affect the computational efficiency and performance of the proposed algorithms. Recently, some new methods with adaptive non-monotonic step sizes were proposed to solve variational inequality problems; see, for instance, [26, 27]. Due to the existence of a class of operators that do not satisfy monotonicity in real applications (e.g., pseudomonotone operators), the methods presented in [17, 18, 19, 20, 24] for solving monotone variational inequalities will not be available in the case where the operators involved are non-monotone. This inspires us to develop algorithms that can solve a wider range of pseudomonotone variational inequalities. Recently, several algorithms were developed to solve pseudomonotone variational inequality problems in Hilbert spaces; see, e.g., [21, 22, 23, 25]. Another aspect of relevance in this study is the convergence speed of the algorithm. Fast iterative algorithms are preferred to fulfill the computing demands of real applications. As one of the acceleration approaches, the inertial idea is widely utilized and explored by academics in optimization and engineering. The essential principle behind inertial techniques is that the next iteration is determined by the sum of the preceding two (or more) iterations. This minor adjustment can significantly enhance the convergence speed of the algorithm used. Recently, many researchers constructed a large number of numerical methods to solve variational inequalities, equilibrium problems, splitting problems, inclusion problems, etc; see, e.g., [28, 29, 30, 31, 32] and the references therein.

Inspired by the preceding research effort as well as some ongoing work in the subject, four inertial projection and contraction approaches for solving variational inequalities in Hilbert spaces are presented in this paper. The following is a summary of our contributions.

(1) Our four algorithms employ a novel non-monotonic step size rule allowing them to work without knowing the prior knowledge of the Lipschitz constant of the operator involved, which is preferable to the fixed-step algorithms presented in [18, 21, 22] for practical applications.
(2) We introduce a new parameter β to modify the projection and contraction method introduced in [11] and the modified subgradient extragradient method proposed in [17]. Numerical results show that the new parameter β has a significant effect on the convergence speed of the methods presented in this paper.
(3) Our four iterative schemes obtain strong convergence in infinite-dimensional Hilbert spaces, which is preferable to the weak convergence results in the literature (e.g., [17, 18, 23, 26]).
(4) The operators involved in our algorithms are pseudomonotone, which is a broader class of operators than monotone operators. Therefore, the four schemes presented in this paper are more useful than the methods for solving monotone variational inequality problems used in the literature (e.g., [17, 18, 19, 20]). In addition, the inertial factor is also added to the proposed algorithms to improve their convergence speed.

The remainder of this paper is structured as follows. The next section covers some of the fundamentals that will be required in the sequel. Section 3 is devoted to presenting and analyzing the convergence of four adaptive iterative techniques for solving variational inequalities. Section 4 provides two simple numerical examples to show the computational efficiency of the suggested methods. A brief summary of the paper is presented in Section 5, the last section.

2. Preliminaries

Let $\mathscr{X}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$. The weak convergence and strong convergence of $\{b_n\}$ to b are represented by $b_n \rightharpoonup b$ and $b_n \to b$, respectively. For each $a,b,c \in \mathscr{H}$, we have the following known inequality and equality

(1) $\|b+a\|^2 \le \|b\|^2 + 2\langle a, b+a\rangle$;
(2) $\|\eta b+\beta a+\lambda c\|^2 = \eta\|b\|^2+\beta\|a\|^2+\lambda\|c\|^2-\eta\beta\|b-a\|^2-\eta\lambda\|b-c\|^2-\beta\lambda\|a-c\|^2$, where $\eta,\beta,\lambda \in [0,1]$ with $\eta+\beta+\lambda=1$.

For each point $b \in \mathscr{H}$, there exists a unique nearest point in $\mathscr{X}$, denoted by $P_{\mathscr{X}}(b)$, such that $P_{\mathscr{X}}(b) = \arg\min\{\|b-a\|, a \in \mathscr{X}\}$. $P_{\mathscr{X}}$ is called the *metric projection* of $\mathscr{H}$ onto $\mathscr{X}$. It is known that $P_{\mathscr{X}}$ has the following basic properties:

$$\langle b - P_{\mathscr{X}}(b), a - P_{\mathscr{X}}(b)\rangle \leq 0, \quad \forall b \in \mathscr{H}, a \in \mathscr{X}, \tag{2.1}$$

and

$$\|P_{\mathscr{X}}(b) - P_{\mathscr{X}}(a)\|^2 \leq \langle P_{\mathscr{X}}(b) - P_{\mathscr{X}}(a), b - a\rangle, \quad \forall b, a \in \mathscr{H}. \tag{2.2}$$

Recall that an operator $Q : \mathscr{H} \to \mathscr{H}$ is said to be

(1) *L-Lipschitz continuous* with $L > 0$ if $\|Qb - Qa\| \leq L\|b - a\|, \forall b, a \in \mathscr{H}$;
(2) *κ-contraction* with $\kappa \in [0, 1)$ if $\|Qb - Qa\| \leq \kappa\|b - a\|, \forall b, a \in \mathscr{H}$;
(3) *monotone* if $\langle Qb - Qa, b - a\rangle \geq 0, \forall b, a \in \mathscr{H}$;
(4) *pseudomonotone* if $\langle Qb, a - b\rangle \geq 0 \Rightarrow \langle Qa, a - b\rangle \geq 0, \forall b, a \in \mathscr{H}$;
(5) *sequentially weakly continuous* if for each sequence $\{b_n\}$ converges weakly to b implies that $\{Qb_n\}$ converges weakly to Qb.

Lemma 2.1 ([33])**.** *Let $\{\lambda_n\}$, $\{\xi_n\}$, and $\{\rho_n\}$ be sequences of nonnegative numbers such that $\lambda_{n+1} \leq \xi_n\lambda_n + \rho_n$, $\forall n \in \mathbb{N}$. If $\{\xi_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$, and $\sum_{n=1}^{\infty}\rho_n < \infty$, then $\lim_{n\to\infty}\lambda_n$ exists.*

Lemma 2.2 ([34])**.** *Let $\{b_n\}$ be a positive sequence, $\{a_n\}$ be a sequence of real numbers and $\{\chi_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty}\chi_n = \infty$. Suppose that $b_{n+1} \leq (1 - \chi_n)b_n + \chi_n a_n$, $\forall n \geq 1$. If $\limsup_{m\to\infty} a_{n_m} \leq 0$ for any subsequence $\{b_{n_m}\}$ of $\{b_n\}$ satisfying $\liminf_{m\to\infty}(b_{n_m+1} - b_{n_m}) \geq 0$, then $\lim_{n\to\infty} b_n = 0$.*

3. Main Results

In this section, we present the four modified inertial projection and contraction methods with a non-monotonic step size criterion for solving pseudomonotone variational inequalities in real Hilbert spaces. They can work well without the prior knowledge of the Lipschitz constant of the operator.

3.1. The Mann-type projection algorithms.

In this subsection, we propose two Mann-type iterative schemes to discover the solutions of variational inequalities. Our methods are inspired by the inertial method, the modified subgradient extragradient method [17], and the Mann-type method. Now we are in a position to state our Algorithm 3.1

The following prerequisites are assumed to be satisfied to study the convergence of Algorithm 3.1.

(C1) The feasible set $\mathscr{X}$ is a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$, and the solution set of the (VIP) is nonempty, i.e., $\Omega \neq \emptyset$.
(C2) The mapping $Q : \mathscr{H} \to \mathscr{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathscr{H}$, and sequentially weakly continuous on $\mathscr{X}$.
(C3) Assume $\lambda_1 > 0$, $\mu \in (0, 1)$, $\{\alpha_n\} \subset [1, \infty)$ with $\lim_{n\to\infty}\alpha_n = 1$, $\{\xi_n\} \subset [1, \infty)$ with $\sum_{n=0}^{\infty}(\xi_n - 1) < \infty$, and $\{\rho_n\} \subset [0, \infty)$ with $\sum_{n=0}^{\infty}\rho_n < \infty$.
(C4) Let $\{\varepsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty}\frac{\varepsilon_n}{\chi_n} = 0$, where $\{\chi_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty}\chi_n = 0$ and $\sum_{n=1}^{\infty}\chi_n = \infty$. Let $\{\tau_n\} \subset (a, b) \subset (0, 1 - \chi_n)$ for some $a > 0, b > 0$.

We first establish a lemma which verifies that the step size criterion (3.2) is valid.

Algorithm 3.1

Initialization: Take $\phi > 0$, $\lambda_1 > 0$, $\eta \in (0, 2/\mu)$, and $\beta \in (\eta/2, 1/\mu)$. Let $b_0, b_1 \in \mathscr{H}$.
Iterative Steps: Calculate the next iteration point b_{n+1} as follows:
Step 1. Compute $a_n = b_n + \phi_n(b_n - b_{n-1})$, where

$$\phi_n = \begin{cases} \min\left\{\dfrac{\varepsilon_n}{\|b_n - b_{n-1}\|}, \phi\right\}, & \text{if } b_n \neq b_{n-1}; \\ \phi, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $g_n = P_{\mathscr{X}}(a_n - \beta\lambda_n Q a_n)$, where the next step size λ_{n+1} is updated by

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\alpha_n \|a_n - g_n\|}{\|Qa_n - Qg_n\|}, \xi_n\lambda_n + \rho_n\right\}, & \text{if } Qa_n \neq Qg_n; \\ \xi_n\lambda_n + \rho_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

If $a_n = g_n$, then stop and $g_n \in \Omega$; otherwise, go to *Step 3*.
Step 3. Compute $d_n = P_{H_n}(a_n - \eta\lambda_n\omega_n Q g_n)$, where

$$H_n = \{b \in \mathscr{H} \mid \langle a_n - \beta\lambda_n Q a_n - g_n, b - g_n\rangle \leq 0\},$$

and

$$\omega_n = \frac{\langle a_n - g_n, \upsilon_n\rangle}{\|\upsilon_n\|^2}, \quad \upsilon_n = a_n - g_n - \beta\lambda_n(Qa_n - Qg_n). \tag{3.3}$$

Step 4. Compute $b_{n+1} = (1 - \chi_n - \tau_n)a_n + \tau_n d_n$.
Set $n := n+1$ and go to *Step 1*.

Lemma 3.1. *Assume that Condition* (C3) *holds. Then the sequence* $\{\lambda_n\}$ *generated by* (3.2) *is well defined and* $\lim_{n\to\infty}\lambda_n$ *exists.*

Proof. Since Q is Lipschitz continuous with $L > 0$ and $\alpha_n \geq 1$, one obtains

$$\frac{\mu\alpha_n\|a_n - g_n\|}{\|Qa_n - Qg_n\|} \geq \frac{\mu\alpha_n\|a_n - g_n\|}{L\|a_n - g_n\|} \geq \frac{\mu}{L}.$$

This together with $\xi_n \geq 1$ and $\chi_n > 0$ yields

$$\lambda_{n+1} = \min\left\{\frac{\mu\alpha_n\|a_n - g_n\|}{\|Qa_n - Qg_n\|}, \xi_n\lambda_n + \rho_n\right\} \geq \min\left\{\frac{\mu}{L}, \lambda_n\right\}.$$

By induction, we have that $\{\lambda_n\}$ has a lower bound $\{\mu/L, \lambda_1\}$. It follows from (3.2) that $\lambda_{n+1} \leq \xi_n\lambda_n + \rho_n$. Combining Lemma 2.1 and Condition (C3), we conclude that $\lim_{n\to\infty}\lambda_n$ exists. That is the desired result. □

Remark 3.1. We show that if $a_n = g_n$ or $\upsilon_n = 0$ in Algorithm 3.1, then $g_n \in \Omega$. By the definition of υ_n and (3.2), one obtains

$$\begin{aligned} \|\upsilon_n\| &\geq \|a_n - g_n\| - \beta\lambda_n\|Qa_n - Qg_n\| \\ &\geq \|a_n - g_n\| - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\|a_n - g_n\| \\ &= \left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\|a_n - g_n\|. \end{aligned} \tag{3.4}$$

It can be easily showed that

$$\|\upsilon_n\| \leq \left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\|a_n-g_n\|. \tag{3.5}$$

Combining (3.4) and (3.5), we have

$$\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\|a_n-g_n\| \leq \|\upsilon_n\| \leq \left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\|a_n-g_n\|.$$

From $\lim_{n\to\infty}\alpha_n=1$ and Lemma 3.1, one sees that

$$\lim_{n\to\infty}\frac{\alpha_n\lambda_n}{\lambda_{n+1}}=1.$$

As a result, there exists a constant n_0 such that $1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}>0$ for all $n\geq n_0$ (noting that $\beta<1/\mu$). Thus we obtain that $a_n=g_n$ if and only if $\upsilon_n=0$. Hence, if $a_n=g_n$ or $\upsilon_n=0$, then $g_n=P_{\mathscr{X}}(g_n-\beta\lambda_n Qg_n)$. This implies that $g_n\in\Omega$ by means of (2.1).

The following lemmas are very helpful in analyzing the convergence of Algorithm 3.1.

Lemma 3.2. *Suppose that Conditions* (C1) *and* (C2) *hold. Let* $\{a_n\}$ *and* $\{g_n\}$ *be two sequences generated by Algorithm 3.1. If there exists a subsequence* $\{a_{n_m}\}$ *of* $\{a_n\}$ *such that* $\{a_{n_m}\}$ *converges weakly to* $c\in\mathscr{H}$ *and* $\lim_{m\to\infty}\|a_{n_m}-g_{n_m}\|=0$*, then* $c\in\Omega$.

Proof. The proof follows the proof in [35, Lemma 3.8] and thus it is omitted. □

Lemma 3.3. *Suppose that Conditions* (C1) *and* (C2) *hold. Let* $\{d_n\}$*,* $\{g_n\}$*, and* $\{a_n\}$ *be three sequences generated by Algorithm 3.1. Then, for every* $b^*\in\Omega$*, there exists* $n_0>0$ *such that*

$$\|d_n-b^*\|^2 \leq \|a_n-b^*\|^2-\|a_n-d_n-\frac{\eta}{\beta}\omega_n\upsilon_n\|^2-\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n-g_n\|^2$$

for all $n\geq n_0$.

Proof. By $b^*\in\Omega\subset\mathscr{X}\subset H_n$ and (2.2), we obtain

$$\begin{aligned}
2\|d_n-b^*\|^2 &= 2\|P_{H_n}(a_n-\eta\lambda_n\omega_n Qg_n)-P_{H_n}(b^*)\|^2\\
&\leq 2\langle d_n-b^*, a_n-\eta\lambda_n\omega_n Qg_n-b^*\rangle\\
&= \|d_n-b^*\|^2+\|a_n-\eta\lambda_n\omega_n Qg_n-b^*\|^2-\|d_n-a_n+\eta\lambda_n\omega_n Qg_n\|^2\\
&= \|d_n-b^*\|^2+\|a_n-b^*\|^2+\eta^2\lambda_n^2\omega_n^2\|Qg_n\|^2-2\langle a_n-b^*,\eta\lambda_n\omega_n Qg_n\rangle\\
&\quad -\|d_n-a_n\|^2-\eta^2\lambda_n^2\omega_n^2\|Qg_n\|^2-2\langle d_n-a_n,\eta\lambda_n\omega_n Qg_n\rangle\\
&= \|d_n-b^*\|^2+\|a_n-b^*\|^2-\|d_n-a_n\|^2-2\langle d_n-b^*,\eta\lambda_n\omega_n Qg_n\rangle,
\end{aligned}$$

which means that

$$\|d_n-b^*\|^2 \leq \|a_n-b^*\|^2-\|d_n-a_n\|^2-2\eta\lambda_n\omega_n\langle d_n-b^*, Qg_n\rangle. \tag{3.6}$$

Using $g_n\in\mathscr{X}$ and $b^*\in\Omega$, one has $\langle Qb^*, g_n-b^*\rangle\geq 0$. This together with the pseudomonotonicity of mapping Q yields $\langle Qg_n, g_n-b^*\rangle\geq 0$, which implies that

$$\langle Qg_n, d_n-b^*\rangle \geq \langle Qg_n, d_n-g_n\rangle. \tag{3.7}$$

Note that $\omega_n > 0$ for all $n \geq n_0$. Indeed, by the definitions of ω_n, υ_n, and (3.2), we have

$$\begin{aligned}\omega_n = \frac{\langle a_n - g_n, \upsilon_n\rangle}{\|\upsilon_n\|^2} &= \frac{\|a_n - g_n\|^2 - \langle a_n - g_n, \beta\lambda_n(Qa_n - Qg_n)\rangle}{\|\upsilon_n\|^2} \\ &\geq \frac{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\|a_n - g_n\|^2}{\|\upsilon_n\|^2}.\end{aligned} \tag{3.8}$$

Note that $1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}} > 0$ for all $n \geq n_0$. Combining (3.5) and (3.8), we obtain

$$\omega_n \geq \frac{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)}{\left(1 + \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2} > 0, \quad \forall n \geq n_0. \tag{3.9}$$

From (3.7) and (3.9) (noting that $\eta \in (0, 2/\mu)$), we deduce that

$$-2\eta\lambda_n\omega_n\langle Qg_n, d_n - b^*\rangle \leq -2\eta\lambda_n\omega_n\langle Qg_n, d_n - g_n\rangle. \tag{3.10}$$

By $d_n \in H_n$ and the definition of H_n, one has $\langle a_n - \beta\lambda_n Qa_n - g_n, d_n - g_n\rangle \leq 0$. This shows that

$$\langle a_n - g_n - \beta\lambda_n(Qa_n - Qg_n), d_n - g_n\rangle \leq \beta\lambda_n\langle Qg_n, d_n - g_n\rangle. \tag{3.11}$$

Using the definitions of υ_n, ω_n, (3.10), and (3.11), we obtain

$$\begin{aligned}-2\eta\lambda_n\omega_n\langle Qg_n, d_n - b^*\rangle &\leq -2\frac{\eta}{\beta}\omega_n\langle \upsilon_n, d_n - g_n\rangle \\ &= -2\frac{\eta}{\beta}\omega_n\langle \upsilon_n, a_n - g_n\rangle + 2\frac{\eta}{\beta}\omega_n\langle \upsilon_n, a_n - d_n\rangle \\ &= -2\frac{\eta}{\beta}\omega_n^2\|\upsilon_n\|^2 + 2\frac{\eta}{\beta}\omega_n\langle \upsilon_n, a_n - d_n\rangle.\end{aligned} \tag{3.12}$$

Now, we estimate $2\frac{\eta}{\beta}\omega_n\langle \upsilon_n, a_n - d_n\rangle$. According to the formula $2ab = a^2 + b^2 - (a-b)^2$, we have

$$2\frac{\eta}{\beta}\omega_n\langle \upsilon_n, a_n - d_n\rangle = \|a_n - d_n\|^2 + \frac{\eta^2}{\beta^2}\omega_n^2\|\upsilon_n\|^2 - \|a_n - d_n - \frac{\eta}{\beta}\omega_n\upsilon_n\|^2. \tag{3.13}$$

It follows from (3.8) that

$$\omega_n\|\upsilon_n\|^2 \geq \left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\|a_n - g_n\|^2.$$

This together with (3.5) implies

$$\omega_n^2\|\upsilon_n\|^2 \geq \left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2 \frac{\|a_n - g_n\|^4}{\|\upsilon_n\|^2} \geq \frac{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n - g_n\|^2. \tag{3.14}$$

Combining (3.6), (3.12), (3.13), and (3.14), we conclude that

$$\begin{aligned}\|d_n - b^*\|^2 \leq \|a_n - b^*\|^2 &- \|a_n - d_n - \frac{\eta}{\beta}\omega_n\upsilon_n\|^2 \\ &- \frac{\eta}{\beta^2}(2\beta - \eta)\frac{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n - g_n\|^2, \ \forall n \geq n_0.\end{aligned}$$

The proof is completed. □

We now prove the strong convergence of Algorithm 3.1.

Theorem 3.1. *Suppose that Condition* (C1)–(C4) *hold. Then the sequence* $\{b_n\}$ *generated by Algorithm 3.1 converges strongly to* $b^* \in \Omega$, *where* $\|b^*\| = \min\{\|c\| : c \in \Omega\}$.

Proof. First, we show that the sequence $\{b_n\}$ is bounded. Indeed, thanks to Lemma 3.3 (noting that $2\beta - \eta > 0$), one obtains

$$\|d_n - b^*\| \le \|a_n - b^*\|, \quad \forall n \ge n_0. \tag{3.15}$$

By the definition of a_n, we have

$$\|a_n - b^*\| \le \|b_n - b^*\| + \chi_n \cdot \frac{\phi_n}{\chi_n}\|b_n - b_{n-1}\|. \tag{3.16}$$

From (3.1), we obtain $\phi_n\|b_n - b_{n-1}\| \le \varepsilon_n$ for all $n \ge 1$, which together with $\lim_{n\to\infty} \frac{\varepsilon_n}{\chi_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\phi_n}{\chi_n}\|b_n - b_{n-1}\| \le \lim_{n\to\infty} \frac{\varepsilon_n}{\chi_n} = 0. \tag{3.17}$$

According to (3.17), we obtain that $\frac{\phi_n}{\chi_n}\|b_n - b_{n-1}\| \to 0$ as $n \to \infty$. Therefore, there exists a constant $S_1 > 0$ such that

$$\frac{\phi_n}{\chi_n}\|b_n - b_{n-1}\| \le S_1, \quad \forall n \ge 1,$$

which combining with (3.15) and (3.16) produces

$$\|d_n - b^*\| \le \|a_n - b^*\| \le \|b_n - b^*\| + \chi_n S_1, \ \forall n \ge n_0. \tag{3.18}$$

By the definition of b_{n+1}, we have

$$\|b_{n+1} - b^*\| \le \|(1 - \chi_n - \tau_n)(a_n - b^*) + \tau_n(d_n - b^*)\| + \chi_n\|b^*\|. \tag{3.19}$$

From (3.15), we obtain

$$\begin{aligned}
&\|(1-\chi_n-\tau_n)(a_n-b^*)+\tau_n(d_n-b^*)\|^2 \\
\le\ & (1-\chi_n-\tau_n)^2\|a_n-b^*\|^2+\tau_n^2\|d_n-b^*\|^2+2(1-\chi_n-\tau_n)\tau_n\|d_n-b^*\|\|a_n-b^*\| \\
\le\ & (1-\chi_n-\tau_n)^2\|a_n-b^*\|^2+\tau_n^2\|a_n-b^*\|^2+2(1-\chi_n-\tau_n)\tau_n\|a_n-b^*\|^2 \\
=\ & (1-\chi_n)^2\|a_n-b^*\|^2,
\end{aligned}$$

which yields

$$\|(1 - \chi_n - \tau_n)(a_n - b^*) + \tau_n(d_n - b^*)\| \le (1 - \chi_n)\|a_n - b^*\|. \tag{3.20}$$

Combining (3.18), (3.19), and (3.20), we have

$$\begin{aligned}
\|b_{n+1} - b^*\| &\le (1-\chi_n)\|a_n - b^*\| + \chi_n\|b^*\| \\
&\le (1-\chi_n)\|b_n - b^*\| + \chi_n(\|b^*\| + S_1) \\
&\le \max\{\|b_n - b^*\|, \|b^*\| + S_1\}, \quad \forall n \ge n_0 \\
&\le \cdots \le \max\{\|b_{n_0} - b^*\|, \|b^*\| + S_1\}.
\end{aligned}$$

That is, the sequence $\{b_n\}$ is bounded, so are $\{a_n\}$, $\{g_n\}$, and $\{d_n\}$.

Using (3.18), we have

$$\begin{aligned}\|a_n-b^*\|^2 &\le (\|b_n-b^*\|+\chi_nS_1)^2\\ &= \|b_n-b^*\|^2+\chi_n(2S_1\|b_n-b^*\|+\chi_nS_1^2)\\ &\le \|b_n-b^*\|^2+\chi_nS_2\end{aligned}\tag{3.21}$$

for some $S_2>0$. By means of Condition (C4) and the definition of b_{n+1}, we obtain

$$\begin{aligned}\|b_{n+1}-b^*\|^2 &= \|(1-\chi_n-\tau_n)(a_n-b^*)+\tau_n(d_n-b^*)+\chi_n(-b^*)\|^2\\ &= (1-\chi_n-\tau_n)\|a_n-b^*\|^2+\tau_n\|d_n-b^*\|^2+\chi_n\|b^*\|^2\\ &\quad -\tau_n(1-\chi_n-\tau_n)\|a_n-d_n\|^2-\chi_n\tau_n\|d_n\|^2-\chi_n(1-\chi_n-\tau_n)\|a_n\|^2\\ &\le (1-\chi_n-\tau_n)\|a_n-b^*\|^2+\tau_n\|d_n-b^*\|^2+\chi_n\|b^*\|^2.\end{aligned}\tag{3.22}$$

Combining (3.21), (3.22), and Lemma 3.3, we have

$$\begin{aligned}\|b_{n+1}-b^*\|^2 &\le (1-\chi_n-\tau_n)\|a_n-b^*\|^2+\tau_n\|a_n-b^*\|^2-\tau_n\|a_n-d_n-\frac{\eta}{\beta}\omega_n\upsilon_n\|^2\\ &\quad -\tau_n\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n-g_n\|^2+\chi_n\|b^*\|^2\\ &\le \|b_n-b^*\|^2-\tau_n\|a_n-d_n-\frac{\eta}{\beta}\omega_n\upsilon_n\|^2+\chi_n(\|b^*\|^2+S_2)\\ &\quad -\tau_n\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n-g_n\|^2,\ \forall n\ge n_0.\end{aligned}$$

Thus,

$$\begin{aligned}&\tau_n\|a_n-d_n-\frac{\eta}{\beta}\omega_n\upsilon_n\|^2+\tau_n\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n-g_n\|^2\\ &\le \|b_n-b^*\|^2-\|b_{n+1}-b^*\|^2+\chi_n(\|b^*\|^2+S_2),\ \forall n\ge n_0.\end{aligned}\tag{3.23}$$

According to the definition of a_n, one has

$$\begin{aligned}\|a_n-b^*\|^2 &\le \|b_n-b^*\|^2+2\phi_n\|b_n-b^*\|\|b_n-b_{n-1}\|+\phi_n^2\|b_n-b_{n-1}\|^2\\ &\le \|b_n-b^*\|^2+3S\phi_n\|b_n-b_{n-1}\|,\end{aligned}\tag{3.24}$$

where $S:=\sup_{n\in\mathbb{N}}\{\|b_n-b^*\|,\phi\|b_n-b_{n-1}\|\}>0$. Set $d_n=(1-\tau_n)a_n+\tau_nd_n$. According to (3.15), we have

$$\|d_n-b^*\|\le(1-\tau_n)\|a_n-b^*\|+\tau_n\|d_n-b^*\|\le\|a_n-b^*\|,\ \forall n\ge n_0.\tag{3.25}$$

Combining (3.24) and (3.25), we have

$$\begin{aligned}\|b_{n+1}-b^*\|^2 &= \|(1-\chi_n)(d_n-b^*)-\chi_n(a_n-d_n)-\chi_n b^*\|^2\\ &\le (1-\chi_n)^2\|d_n-b^*\|^2-2\chi_n\langle a_n-d_n+b^*,b_{n+1}-b^*\rangle\\ &= (1-\chi_n)^2\|d_n-b^*\|^2+2\chi_n\langle a_n-d_n,b^*-b_{n+1}\rangle+2\chi_n\langle b^*,b^*-b_{n+1}\rangle\\ &\le (1-\chi_n)\|d_n-b^*\|^2+2\chi_n\|a_n-d_n\|\|b_{n+1}-b^*\|+2\chi_n\langle b^*,b^*-b_{n+1}\rangle\\ &\le (1-\chi_n)\|b_n-b^*\|^2+\chi_n\Big[2\tau_n\|a_n-d_n\|\|b_{n+1}-b^*\|\\ &\quad +2\langle b^*,b^*-b_{n+1}\rangle+\frac{3S\phi_n}{\chi_n}\|b_n-b_{n-1}\|\Big],\ \forall n\ge n_0.\end{aligned} \tag{3.26}$$

Finally, we need to show that $\{\|b_n-b^*\|\}$ converges to zero. Throughout this paper, we always suppose that $\{\|b_{n_m}-b^*\|\}$ is a subsequence of $\{\|b_n-b^*\|\}$ such that $\liminf_{m\to\infty}(\|b_{n_m+1}-b^*\|-\|b_{n_m}-b^*\|)\ge 0$. Then,

$$\begin{aligned}&\liminf_{m\to\infty}\left(\|b_{n_m+1}-b^*\|^2-\|b_{n_m}-b^*\|^2\right)\\ &=\liminf_{m\to\infty}\left[(\|b_{n_m+1}-b^*\|-\|b_{n_m}-b^*\|)(\|b_{n_m+1}-b^*\|+\|b_{n_m}-b^*\|)\right]\ge 0.\end{aligned}$$

Combining (3.23), Condition (C4), $\eta\in(0,2/\mu)$, and $\beta\in(\eta/2,1/\mu)$, we have

$$\begin{aligned}&\limsup_{m\to\infty}\left\{\tau_{n_m}\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_{n_m}\lambda_{n_m}}{\lambda_{n_m+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_{n_m}\lambda_{n_m}}{\lambda_{n_m+1}}\right)^2}\|a_{n_m}-g_{n_m}\|^2+\tau_{n_m}\|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|^2\right\}\\ &\le\limsup_{m\to\infty}\left[\|b_{n_m}-b^*\|^2-\|b_{n_m+1}-b^*\|^2\right]+\limsup_{m\to\infty}\chi_{n_m}(\|b^*\|^2+S_2)\\ &=-\liminf_{m\to\infty}\left[\|b_{n_m+1}-b^*\|^2-\|b_{n_m}-b^*\|^2\right]\le 0,\end{aligned}$$

which yields that

$$\lim_{m\to\infty}\|g_{n_m}-a_{n_m}\|=0\ \text{ and }\ \lim_{m\to\infty}\|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|=0.$$

From the definition of ω_{n_m}, we obtain

$$\begin{aligned}\|a_{n_m}-d_{n_m}\| &\le \|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|+\frac{\eta}{\beta}\omega_{n_m}\|\upsilon_{n_m}\|\\ &= \|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|+\frac{\eta}{\beta}\frac{\langle a_{n_m}-g_{n_m},\upsilon_{n_m}\rangle}{\|\upsilon_{n_m}\|}\\ &\le \|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|+\frac{\eta}{\beta}\|a_{n_m}-g_{n_m}\|.\end{aligned}$$

Hence we have that $\lim_{m\to\infty}\|d_{n_m}-a_{n_m}\|=0$. This together with the boundedness of $\{b_n\}$ yields

$$\lim_{m\to\infty}\tau_{n_m}\|a_{n_m}-d_{n_m}\|\|b_{n_m+1}-b^*\|=0. \tag{3.27}$$

From Condition (C4) and (3.17), we obtain

$$\|b_{n_m+1}-a_{n_m}\|=\chi_{n_m}\|a_{n_m}\|+\tau_{n_m}\|d_{n_m}-a_{n_m}\|\to 0 \text{ as } m\to\infty,$$

and

$$\|b_{n_m} - a_{n_m}\| = \chi_{n_m} \cdot \frac{\phi_{n_m}}{\chi_{n_m}} \|b_{n_m} - b_{n_m-1}\| \to 0 \text{ as } m \to \infty.$$

From the above facts, we conclude that

$$\|b_{n_m+1} - b_{n_m}\| \leq \|b_{n_m+1} - a_{n_m}\| + \|a_{n_m} - b_{n_m}\| \to 0 \text{ as } m \to \infty. \tag{3.28}$$

Since the sequence $\{b_{n_m}\}$ is bounded, there exists a subsequence $\{b_{n_{m_j}}\}$ of $\{b_{n_m}\}$ such that $b_{n_{m_j}} \rightharpoonup c$ when $j \to \infty$. Furthermore,

$$\limsup_{m\to\infty} \langle b^*, b^* - b_{n_m}\rangle = \lim_{j\to\infty} \langle b^*, b^* - b_{n_{m_j}}\rangle = \langle b^*, b^* - c\rangle. \tag{3.29}$$

We have that $a_{n_m} \rightharpoonup c$ since $\|b_{n_m} - a_{n_m}\| \to 0$. This combining with $\lim_{m\to\infty} \|a_{n_m} - g_{n_m}\| = 0$ and Lemma 3.2 implies that $c \in \Omega$. By (2.1), (3.29) and the definition of b^*, we have

$$\limsup_{m\to\infty} \langle b^*, b^* - b_{n_m}\rangle = \langle b^*, b^* - c\rangle \leq 0. \tag{3.30}$$

From (3.28) and (3.30), one obtains

$$\limsup_{m\to\infty} \langle b^*, b^* - b_{n_m+1}\rangle \leq \limsup_{m\to\infty} \langle b^*, b^* - b_{n_m}\rangle \leq 0. \tag{3.31}$$

Combining (3.17), (3.26), (3.27), (3.31), and Lemma 2.2, we have that $b_n \to b^*$ as $n \to \infty$. That is the required conclusion. □

Next, we introduce a new modified inertial projection and contraction algorithm that is inspired by the Algorithm 3.1 of Gibali et al. [19] to find the solutions of pseudomonotone variational inequalities in infinite-dimensional Hilbert spaces

Algorithm 3.2

Initialization: Take $\phi > 0$, $\lambda_1 > 0$, $\eta \in (0, 2)$, and $\beta \in (0, 1/\mu)$. Let $b_0, b_1 \in \mathscr{H}$.
Iterative Steps: Calculate the next iteration point b_{n+1} as follows:

$$\begin{cases} a_n = b_n + \phi_n (b_n - b_{n-1}), \\ g_n = P_{\mathscr{X}} (a_n - \beta \lambda_n Q a_n), \\ d_n = a_n - \eta \omega_n \upsilon_n, \\ b_{n+1} = (1 - \chi_n - \tau_n) a_n + \tau_n d_n, \end{cases}$$

where $\{\phi_n\}$, $\{\lambda_n\}$, and $\{\omega_n\}$ are defined in (3.1), (3.2), and (3.3), respectively.

The following lemma is critical in analyzing the convergence of Algorithm 3.2.

Lemma 3.4. *Assume Condition* (C1) *and* (C2) *hold. Let* $\{a_n\}$, $\{g_n\}$, *and* $\{d_n\}$ *be three sequences created by Algorithm 3.2. Then*

$$\|d_n - b^*\|^2 \leq \|a_n - b^*\|^2 - \frac{2-\eta}{\eta} \|a_n - d_n\|^2, \ \forall b^* \in \Omega,$$

and

$$\|a_n - g_n\|^2 \leq \left[\frac{\left(1 + \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)}{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\eta} \right]^2 \|a_n - d_n\|^2.$$

Proof. From the definition of d_n, one sees that

$$\begin{aligned}\|d_n - b^*\|^2 &= \|a_n - \eta\omega_n\upsilon_n - b^*\|^2 \\ &= \|a_n - b^*\|^2 - 2\eta\omega_n\langle a_n - b^*, \upsilon_n\rangle + \eta^2\omega_n^2\|\upsilon_n\|^2.\end{aligned} \tag{3.32}$$

Combining (3.2) and (3.3), we obtain

$$\begin{aligned}\langle a_n - b^*, \upsilon_n\rangle &= \langle a_n - g_n, \upsilon_n\rangle + \langle g_n - b^*, \upsilon_n\rangle \\ &= \langle a_n - g_n, \upsilon_n\rangle + \langle g_n - b^*, a_n - g_n - \beta\lambda_n(Qa_n - Qg_n)\rangle.\end{aligned} \tag{3.33}$$

According to $g_n = P_{\mathscr{X}}(a_n - \beta\lambda_n Qa_n)$ and (2.1), we obtain

$$\langle a_n - g_n - \beta\lambda_n Qa_n, g_n - b^*\rangle \geq 0. \tag{3.34}$$

Using $b^* \in \Omega$ and $g_n \in \mathscr{X}$, we have that $\langle Qb^*, g_n - b^*\rangle \geq 0$. This combining with the pseudomonotonicity of mapping Q yields that

$$\langle Qg_n, g_n - b^*\rangle \geq 0. \tag{3.35}$$

By (3.33), (3.34), and (3.35), we obtain

$$\langle a_n - b^*, \upsilon_n\rangle \geq \langle a_n - g_n, \upsilon_n\rangle. \tag{3.36}$$

By the definition of d_n, one obtains $d_n - a_n = \eta\omega_n\upsilon_n$. From the definition of ω_n, we have $\langle a_n - g_n, \upsilon_n\rangle = \omega_n\|\upsilon_n\|^2$. Combining (3.32) and (3.36), we obtain

$$\begin{aligned}\|d_n - b^*\|^2 &\leq \|a_n - b^*\|^2 - 2\eta\omega_n\langle a_n - g_n, \upsilon_n\rangle + \eta^2\omega_n^2\|\upsilon_n\|^2 \\ &= \|a_n - b^*\|^2 - 2\eta\omega_n^2\|\upsilon_n\|^2 + \eta^2\omega_n^2\|\upsilon_n\|^2 \\ &= \|a_n - b^*\|^2 - \frac{2-\eta}{\eta}\|\eta\omega_n\upsilon_n\|^2 \\ &= \|a_n - b^*\|^2 - \frac{2-\eta}{\eta}\|a_n - d_n\|^2.\end{aligned}$$

On the other hand, by the definition of d_n and (3.14), we deudce that

$$\|d_n - a_n\|^2 = \eta^2\omega_n^2\|\upsilon_n\|^2 \geq \eta^2\frac{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1 + \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n - g_n\|^2.$$

Thus we obtain

$$\|a_n - g_n\|^2 \leq \left[\frac{\left(1 + \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)}{\left(1 - \frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)\eta}\right]^2\|a_n - d_n\|^2.$$

The proof is complete. □

Theorem 3.2. *Assume that Condition* (C1)–(C4) *hold. Then the sequence* $\{b_n\}$ *generated by Algorithm 3.2 converges strongly to* $b^* \in \Omega$*, where* $\|b^*\| = \min\{\|c\| : c \in \Omega\}$.

Proof. Thanks to Lemma 3.4 and $\eta \in (0,2)$, we obtain

$$\|d_n - b^*\| \leq \|a_n - b^*\|, \quad \forall n \geq 1. \tag{3.37}$$

We find that the sequences $\{b_n\}$, $\{a_n\}$, $\{g_n\}$, and $\{d_n\}$ are bounded by using the same facts as stated in Theorem 3.1. Combining (3.21), (3.22), and Lemma 3.4, we obtain

$$\begin{aligned}\|b_{n+1}-b^*\|^2 &\le (1-\chi_n-\tau_n)\|a_n-b^*\|^2+\tau_n\|a_n-b^*\|^2-\tau_n\frac{2-\eta}{\eta}\|a_n-d_n\|^2+\chi_n\|b^*\|^2\\ &\le \|b_n-b^*\|^2-\tau_n\frac{2-\eta}{\eta}\|a_n-d_n\|^2+\chi_n(\|b^*\|^2+S_2).\end{aligned}$$

Thus we have

$$\tau_n\frac{2-\eta}{\eta}\|a_n-d_n\|^2 \le \|b_n-b^*\|^2-\|b_{n+1}-b^*\|^2+\chi_n(\|b^*\|^2+S_2). \tag{3.38}$$

Using the same facts as stated in Theorem 3.1, we have that (3.26) holds. Combining (3.38) and Condition (C4), one obtains

$$\begin{aligned}\limsup_{m\to\infty}\tau_{n_m}\frac{2-\eta}{\eta}\|a_{n_m}-d_{n_m}\|^2 &\le \limsup_{m\to\infty}\left[\|b_{n_m}-b^*\|^2-\|b_{n_m+1}-b^*\|^2+\chi_{n_m}(\|b^*\|^2+S_2)\right]\\ &\le 0,\end{aligned}$$

which means that $\lim_{m\to\infty}\|d_{n_m}-a_{n_m}\|=0$. We find that $\lim_{m\to\infty}\|g_{n_m}-a_{n_m}\|=0$ by means of Lemma 3.4. As claimed in Theorem 3.1, one has the same result as (3.27)–(3.31). Thus we obtain that $b_n\to b^*$ as $n\to\infty$. □

3.2. **The viscosity-type projection algorithms.** Next, we introduce two viscosity-based solutions to address variational inequalities. The viscosity version of the proposed Algorithm 3.1 is depicted in Algorithm 3.3 below.

Algorithm 3.3

Initialization: Take $\phi>0$, $\lambda_1>0$, $\eta\in(0,2/\mu)$, and $\beta\in(\eta/2,1/\mu)$. Let $b_0,b_1\in\mathscr{H}$.
Iterative Steps: Calculate the next iteration point b_{n+1} as follows:

$$\begin{cases} a_n=b_n+\phi_n(b_n-b_{n-1}),\\ g_n=P_{\mathscr{X}}(a_n-\beta\lambda_nQa_n),\\ d_n=P_{H_n}(a_n-\eta\lambda_n\omega_nQg_n),\\ H_n=\{b\in\mathscr{H}\mid\langle a_n-\beta\lambda_nQa_n-g_n,b-g_n\rangle\le0\},\\ b_{n+1}=\chi_nf(d_n)+(1-\chi_n)d_n,\end{cases}$$

where $\{\phi_n\}$, $\{\lambda_n\}$, and $\{\omega_n\}$ are defined in (3.1), (3.2), and (3.3), respectively.

Theorem 3.3. *Assume that Condition* (C1)–(C3) *and* (C5) *hold. Then the sequence* $\{b_n\}$ *generated by Algorithm 3.3 converges strongly to* $b^*\in\Omega$*, where* $b^*=P_\Omega(f(b^*))$.

(C5) *Let* $f:\mathscr{H}\to\mathscr{H}$ *be a* κ*-contraction mapping with* $\kappa\in[0,1)$*, and let* $\{\varepsilon_n\}$ *be a positive sequence such that* $\lim_{n\to\infty}\frac{\varepsilon_n}{\chi_n}=0$*, where* $\{\chi_n\}\subset(0,1)$ *satisfies* $\lim_{n\to\infty}\chi_n=0$ *and* $\sum_{n=1}^{\infty}\chi_n=\infty$.

Proof. It follows from the definition of b_{n+1} and (3.18) that

$$
\begin{aligned}
\|b_{n+1}-b^*\| &\leq \chi_n\|f(d_n)-f(b^*)\|+\chi_n\|f(b^*)-b^*\|+(1-\chi_n)\|d_n-b^*\| \\
&\leq (1-(1-\kappa)\chi_n)\|b_n-b^*\|+(1-\kappa)\chi_n\frac{S_1+\|f(b^*)-b^*\|}{1-\kappa},\ \forall n\geq n_0 \\
&\leq \max\left\{\|b_{n_0}-b^*\|,\frac{S_1+\|f(b^*)-b^*\|}{1-\kappa}\right\}.
\end{aligned}
$$

This means that $\{b_n\}$ is bounded. Hence, $\{a_n\}$, $\{g_n\}$, $\{d_n\}$, and $\{f(d_n)\}$ are also bounded. Combining (3.21) and Lemma 3.3, we have

$$
\begin{aligned}
&\|b_{n+1}-b^*\|^2 \\
&\leq \chi_n(\|d_n-b^*\|+\|f(b^*)-b^*\|)^2+(1-\chi_n)\|d_n-b^*\|^2 \\
&= \chi_n\|d_n-b^*\|^2+(1-\chi_n)\|d_n-b^*\|^2+\chi_n\left(2\|d_n-b^*\|\|f(b^*)-b^*\|+\|f(b^*)-b^*\|^2\right) \\
&\leq \|d_n-b^*\|^2+\chi_nS_3 \\
&\leq \|b_n-b^*\|^2-\|a_n-d_n-\frac{\eta}{\beta}\omega_n\upsilon_n\|^2 \\
&\quad -\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n-g_n\|^2+\chi_nS_4,\ \forall n\geq n_0,
\end{aligned}
$$

where $S_3:=\max\{2\|d_n-b^*\|\|f(b^*)-b^*\|+\|f(b^*)-b^*\|^2\}$ and $S_4:=S_2+S_3$. Hence

$$
\begin{aligned}
&\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_n\lambda_n}{\lambda_{n+1}}\right)^2}\|a_n-g_n\|^2+\|a_n-d_n-\frac{\eta}{\beta}\omega_n\upsilon_n\|^2 \\
&\leq \|b_n-b^*\|^2-\|b_{n+1}-b^*\|^2+\chi_nS_4,\ \forall n\geq n_0.
\end{aligned} \tag{3.39}
$$

From (3.15) and (3.24), we have

$$
\begin{aligned}
\|b_{n+1}-b^*\|^2 &= \|\chi_n(f(d_n)-f(b^*))+(1-\chi_n)(d_n-b^*)+\chi_n(f(b^*)-b^*)\|^2 \\
&\leq \chi_n\kappa\|d_n-b^*\|^2+(1-\chi_n)\|d_n-b^*\|^2+2\chi_n\langle f(b^*)-b^*,b_{n+1}-b^*\rangle \\
&\leq (1-(1-\kappa)\chi_n)\|b_n-b^*\|^2+(1-\kappa)\chi_n\cdot\left[\frac{3S}{1-\kappa}\cdot\frac{\phi_n}{\chi_n}\|b_n-b_{n-1}\|\right. \\
&\quad \left.+\frac{2}{1-\kappa}\langle f(b^*)-b^*,b_{n+1}-b^*\rangle\right],\ \forall n\geq n_0.
\end{aligned} \tag{3.40}
$$

Finally, we show that $\{\|b_n-b^*\|\}$ converges to zero. From (3.39), Condition (C5), $\eta\in(0,2/\mu)$, and $\beta\in(\eta/2,1/\mu)$, we have

$$
\begin{aligned}
&\limsup_{m\to\infty}\left\{\frac{\eta}{\beta^2}(2\beta-\eta)\frac{\left(1-\frac{\beta\mu\alpha_{n_m}\lambda_{n_m}}{\lambda_{n_m+1}}\right)^2}{\left(1+\frac{\beta\mu\alpha_{n_m}\lambda_{n_m}}{\lambda_{n_m+1}}\right)^2}\|a_{n_m}-g_{n_m}\|^2+\|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|^2\right\} \\
&\leq \limsup_{m\to\infty}\left[\|b_{n_m}-b^*\|^2-\|b_{n_m+1}-b^*\|^2+\chi_{n_m}S_4\right]\leq 0,
\end{aligned}
$$

which indicates that

$$\lim_{m\to\infty}\|g_{n_m}-a_{n_m}\|=0 \text{ and } \lim_{m\to\infty}\|a_{n_m}-d_{n_m}-\frac{\eta}{\beta}\omega_{n_m}\upsilon_{n_m}\|=0.$$

As claimed in Theorem 3.1, one can show that $\lim_{m\to\infty}\|d_{n_m}-a_{n_m}\|=0$. Combining Condition (C5) and (3.17), we have

$$\|b_{n_m+1}-d_{n_m}\|=\chi_{n_m}\|d_{n_m}-f(b_{n_m})\|\to 0 \text{ as } m\to\infty,$$

and

$$\|b_{n_m}-a_{n_m}\|=\chi_{n_m}\cdot\frac{\phi_{n_m}}{\chi_{n_m}}\|b_{n_m}-b_{n_m-1}\|\to 0 \text{ as } m\to\infty.$$

Therefore,

$$\|b_{n_m+1}-b_{n_m}\|\le\|b_{n_m+1}-d_{n_m}\|+\|d_{n_m}-a_{n_m}\|+\|a_{n_m}-b_{n_m}\|\to 0 \text{ as } m\to\infty. \tag{3.41}$$

Since $\{b_{n_m}\}$ is bounded, there exists a subsequence $\{b_{n_{m_j}}\}$ of $\{b_{n_m}\}$ such that $b_{n_{m_j}}\rightharpoonup c$ as $j\to\infty$. In addition,

$$\begin{aligned}\limsup_{m\to\infty}\langle f(b^*)-b^*,b_{n_m}-b^*\rangle &= \lim_{j\to\infty}\langle f(b^*)-b^*,b_{n_{m_j}}-b^*\rangle\\ &=\langle f(b^*)-b^*,c-b^*\rangle.\end{aligned} \tag{3.42}$$

We have that $a_{n_m}\rightharpoonup c$ as $\|b_{n_m}-a_{n_m}\|\to 0$, which combining with $\lim_{m\to\infty}\|a_{n_m}-g_{n_m}\|=0$ and Lemma 3.2 yields that $c\in\Omega$. Using (2.1), (3.42), and the definition of b^*, we have

$$\limsup_{m\to\infty}\langle f(b^*)-b^*,b_{n_m}-b^*\rangle=\langle f(b^*)-b^*,c-b^*\rangle\le 0. \tag{3.43}$$

By (3.41) and (3.43), one sees that

$$\limsup_{m\to\infty}\langle f(b^*)-b^*,b_{n_m+1}-b^*\rangle\le\limsup_{m\to\infty}\langle f(b^*)-b^*,b_{n_m}-b^*\rangle\le 0. \tag{3.44}$$

From (3.17), (3.40), (3.44), and Lemma 2.2, we have that $b_n\to b^*$ as $n\to\infty$. □

The final iterative technique described in this study is depicted in Algorithm 3.4 below, which is also a viscosity version of the suggested Algorithm 3.2.

Algorithm 3.4

Initialization: Take $\phi>0$, $\lambda_1>0$, $\eta\in(0,2)$, and $\beta\in(0,1/\mu)$. Let $b_0,b_1\in\mathscr{H}$.
Iterative Steps: Calculate the next iteration point b_{n+1} as follows:

$$\begin{cases} a_n=b_n+\phi_n(b_n-b_{n-1}),\\ g_n=P_{\mathscr{X}}(a_n-\beta\lambda_n Qa_n),\\ d_n=a_n-\eta\omega_n\upsilon_n,\\ b_{n+1}=\chi_n f(d_n)+(1-\chi_n)d_n,\end{cases}$$

where $\{\phi_n\}$, $\{\lambda_n\}$, and $\{\omega_n\}$ are defined in (3.1), (3.2), and (3.3), respectively.

Theorem 3.4. *Assume that Condition* (C1)–(C3) *and* (C5) *hold. Then the sequence* $\{b_n\}$ *formed by Algorithm 3.4 converges strongly to* $b^*\in\Omega$*, where* $b^*=P_\Omega(f(b^*))$.

Proof. We can obtain that $\{b_n\}$, $\{a_n\}$, $\{g_n\}$, $\{d_n\}$, and $\{f(d_n)\}$ are bounded by using the same arguments as declared in Theorem 3.3. Combining Lemma 3.4 and (3.21), we have

$$\frac{2-\eta}{\eta}\|a_n-d_n\|^2 \leq \|b_n-b^*\|^2-\|b_{n+1}-b^*\|^2+\chi_n S_4, \tag{3.45}$$

where S_4 is defined in (3.39). Furthermore, we can derive (3.40) by using the same facts as declared in Theorem 3.3. Using Condition (C5) and (3.45), one obtains

$$\limsup_{m\to\infty}\frac{2-\eta}{\eta}\|a_{n_m}-d_{n_m}\|^2 \leq 0,$$

which implies that $\lim_{m\to\infty}\|d_{n_m}-a_{n_m}\|=0$. This together with Lemma 3.4 yields that

$$\lim_{m\to\infty}\|g_{n_m}-a_{n_m}\|=0.$$

As claimed in Theorem 3.3, we can obtain the same facts as (3.41)–(3.44). Hence we deduce that $b_n \to b^*$ as $n\to\infty$. □

4. Computational Experiments

We provide two numerical examples occurring in finite and infinite dimensional spaces to demonstrate the efficiency and computational advantages of the suggested approaches and compare them with the known ones in [25]. All the codes written in this section are realized using MATLAB 2018a on an Intel(R) Core(TM) i5-8250S CPU @1.60 GHz personal computer with RAM 8.00 GB.

4.1. A finite dimensional example.

The first example has been considered by many researchers.

Example 4.1. Give a linear operator $Q:\mathbb{R}^m\to\mathbb{R}^m$ as follows:

$$Q(b)=Gb+g,$$

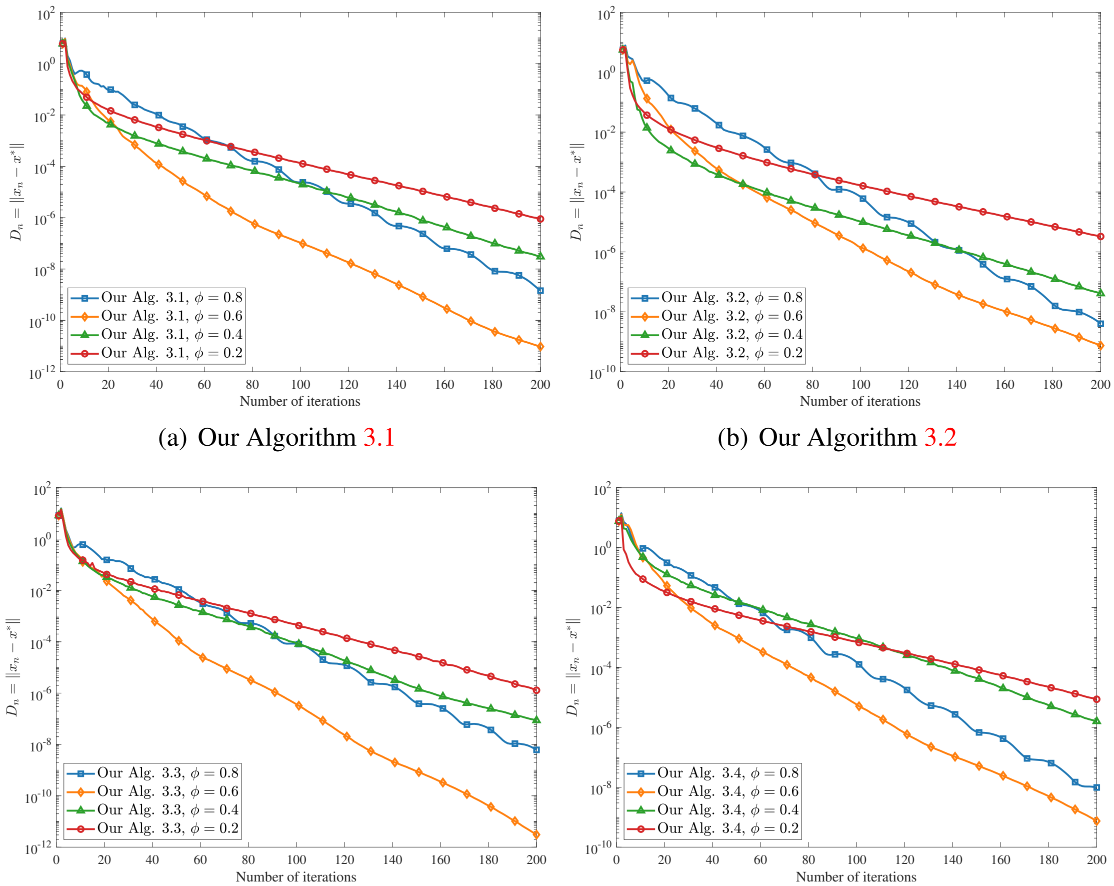
where $g\in\mathbb{R}^m$, $G=BB^{\mathsf{T}}+S+E$, $B\in\mathbb{R}^{m\times m}$, $S\in\mathbb{R}^{m\times m}$ is skew-symmetric, and $E\in\mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are non-negative. The feasible set $\mathscr{X}$ is a box constraint with the form $\mathscr{X}=[-2,5]^m$. It is easy to show that Q is monotone, Lipschitz continuous, and its Lipschitz constant $L=\|G\|$. In this numerical example, all entries of B,S are generated randomly in $[-2,2]$, E is generated randomly in $[0,2]$, and $g=\mathbf{0}$. The solution set of the (VIP) is $b^*=\{\mathbf{0}\}$. The maximum number of iterations of 200 as a common stopping criterion and the initial values $b_0=b_1$ are randomly generated by *rand(m,1)* in MATLAB. We use $D_n=\|b_n-b^*\|$ to measure the n-th iteration error of all algorithms.

4.1.1. *Sensitivity analysis of parameters.*

We test the performance of the proposed algorithms under different parameters. Specifically, we consider the following three cases.

Case 1*: Compare* ϕ. Pick $\eta=1.5$, $\chi_n=1/(n+1)$, $\tau_n=0.8(1-\chi_n)$, $f(b)=0.1b$, $\phi=\{0.2,0.4,0.6,0.8\}$, $\varepsilon_n=100/(n+1)^2$, $\beta=0.8$, $\lambda_1=0.6$, $\mu=0.6$, $\alpha_n=1+1/n$, $\xi_n=1+1/(n+1)^{1.1}$, and $\rho_n=1/(n+1)^{1.1}$ for the proposed Algorithms 3.1–3.4. The numerical behavior of the proposed algorithms with different parameters ϕ is illustrated in Fig. 1. The information shown in Fig. 1 tells us that the inertial factor plays a significant role in the convergence of the proposed methods.

Case 2*: Compare* β. Take $\eta=1.5$, $\chi_n=1/(n+1)$, $\tau_n=0.8(1-\chi_n)$, $f(b)=0.1b$, $\phi=0.6$, $\varepsilon_n=100/(n+1)^2$, $\beta=\{0.8,0.9,1.0,1.1\}$, $\lambda_1=0.6$, $\mu=0.6$, $\alpha_n=1+1/n$, $\xi_n=1+1/(n+$



(a) Our Algorithm 3.1 (b) Our Algorithm 3.2

(c) Our Algorithm 3.3 (d) Our Algorithm 3.4

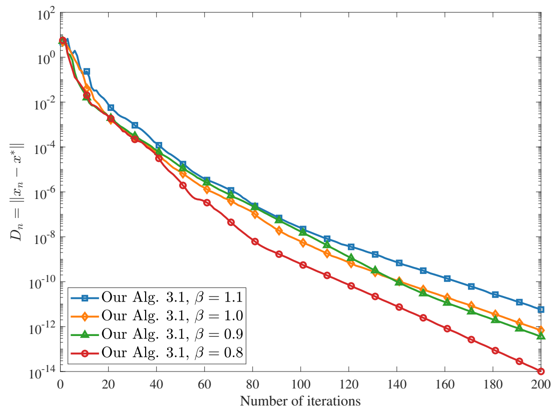
FIGURE 1. The behavior of our algorithms with different ϕ in Example 4.1 ($m = 20$)

$1)^{1.1}$, and $\rho_n = 1/(n+1)^{1.1}$ for the proposed Algorithms 3.1–3.4. The numerical performance of the proposed algorithms with different parameters β is shown in Fig. 2. It can be seen from Fig. 2 that the suggested approaches have a superior numerical performance when the appropriate parameter β is chosen.

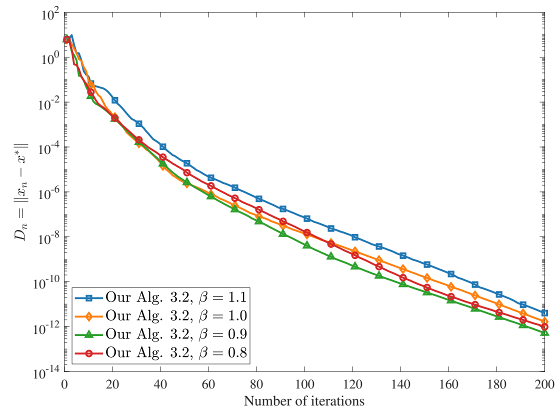
Case 3: *Compare* λ_n. Take $\eta = 1.5$, $\chi_n = 1/(n+1)$, $\tau_n = 0.8(1-\chi_n)$, $f(b) = 0.1b$, $\phi = 0.6$, $\varepsilon_n = 100/(n+1)^2$, $\beta = 0.8$, $\lambda_1 = 0.6$, and $\mu = 0.6$ for the proposed Algorithms 3.1–3.4. We consider the impact of different parameter choices in the step size criterion (3.2) on the proposed algorithms. The numerical behavior of the proposed algorithms with different step size λ_n is expressed in Fig. 3. The findings in Fig. 3 show that utilizing the non-monotonic step size criterion (orange, green, and purple lines in Fig. 3) resulted in faster convergence and accuracy than using the non-increasing step size criterion (blue and red lines in Fig. 3).

4.1.2. *Comparison with known algorithms.* To end this example, we compare the proposed algorithms with the Algorithms 3.1, 3.3, 3.4 and 3.6 presented by Tan, Li and Cho [25] (shortly, TLC Alg. 3.1, TLC Alg. 3.3, TLC Alg. 3.4 and TLC Alg. 3.6). The parameters of all algorithms are set as follows.

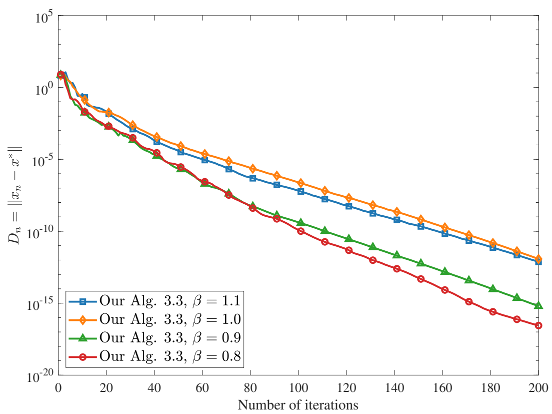
- Take $\phi = 0.6$, $\varepsilon_n = 100/(n+1)^2$, $\eta = 1.5$, $\chi_n = 1/(n+1)$, $\tau_n = 0.8(1-\chi_n)$, and $f(b) = 0.1b$ for all algorithms.



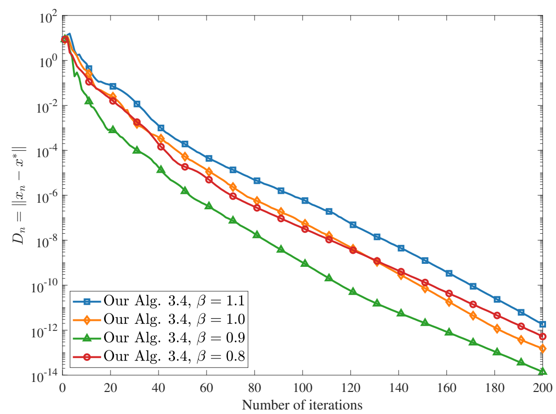
(a) Our Algorithm 3.1



(b) Our Algorithm 3.2



(c) Our Algorithm 3.3



(d) Our Algorithm 3.4

FIGURE 2. The behavior of our algorithms with different β in Example 4.1 ($m = 20$)

- Choose $\beta = 0.8$, $\lambda_1 = 0.6$, $\mu = 0.6$, $\alpha_n = 1 + 1/n$, $\xi_n = 1 + 1/(n+1)^{1.1}$, and $\rho_n = 1/(n+1)^{1.1}$ for the proposed Algorithms 3.1–3.4.
- Pick $\delta = 2$, $\zeta = 0.5$, and $\phi = 0.6$ for TLC Alg. 3.1, TLC Alg. 3.3, TLC Alg. 3.4 and TLC Alg. 3.6.

The performance and numerical results of all algorithms with three dimensions are shown in Fig. 4 and Table 1. We find that the proposed methods have a higher accuracy and faster convergence than the numerical methods introduced by Tan et al. [25], and this result is independent of the size of the dimension.

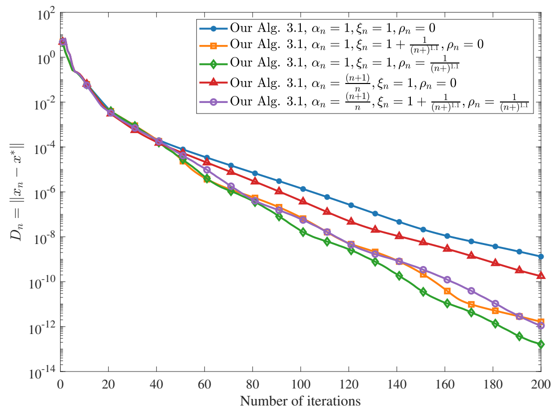
4.2. **An infinite dimensional example.**

Example 4.2. Let $\mathscr{H} = L^2([0,1])$ be an infinite-dimensional Hilbert space with inner product

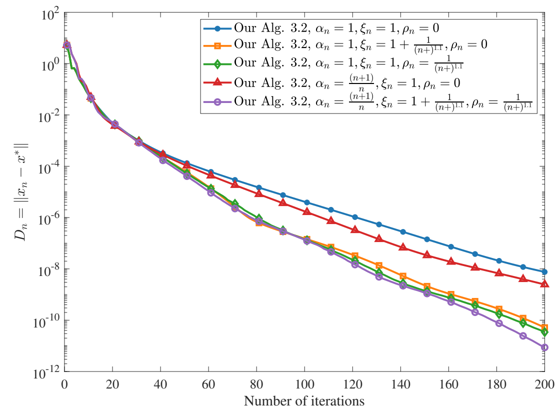
$$\langle b, a \rangle = \int_0^1 b(t)a(t)\,\mathrm{d}t, \quad \forall b, a \in \mathscr{H},$$

and induced norm

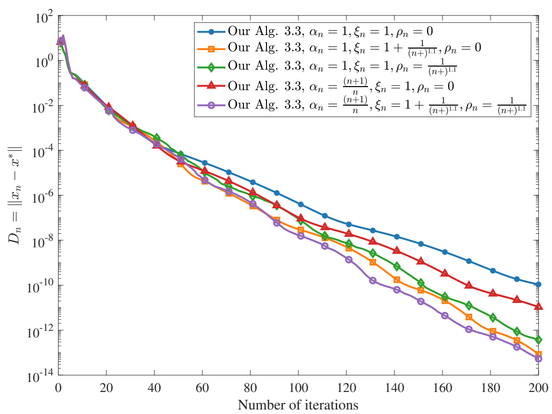
$$\|b\| = \left(\int_0^1 |b(t)|^2\,\mathrm{d}t \right)^{1/2}, \quad \forall b \in \mathscr{H}.$$



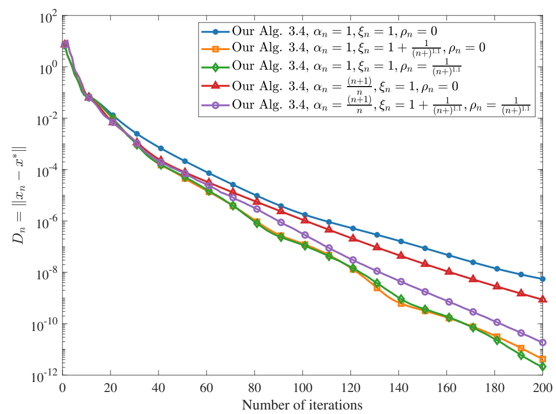
(a) Our Algorithm 3.1



(b) Our Algorithm 3.2

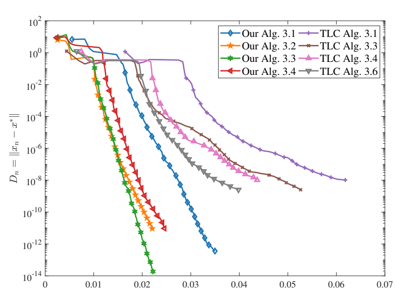


(c) Our Algorithm 3.3

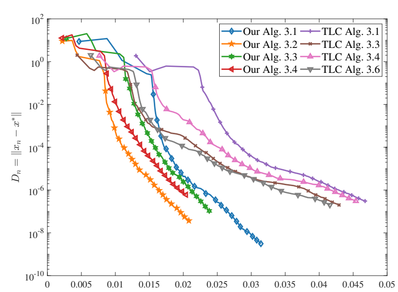


(d) Our Algorithm 3.4

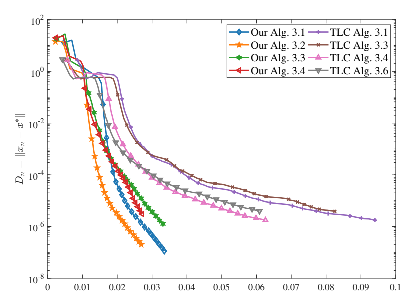
FIGURE 3. The behavior of our algorithms with different λ_n in Example 4.1 ($m = 20$)



(a) $m = 20$



(b) $m = 50$



(c) $m = 100$

FIGURE 4. The behavior of all algorithms with different dimensions for Example 4.1

Assume that r and R are two positive real numbers such that $R/(k+1) < r/k < r < R$ for some $k > 1$. Let $\mathscr{X}$ be defined by

$$\mathscr{X} = \{b \in \mathscr{H} : \|b\| \le r\},$$

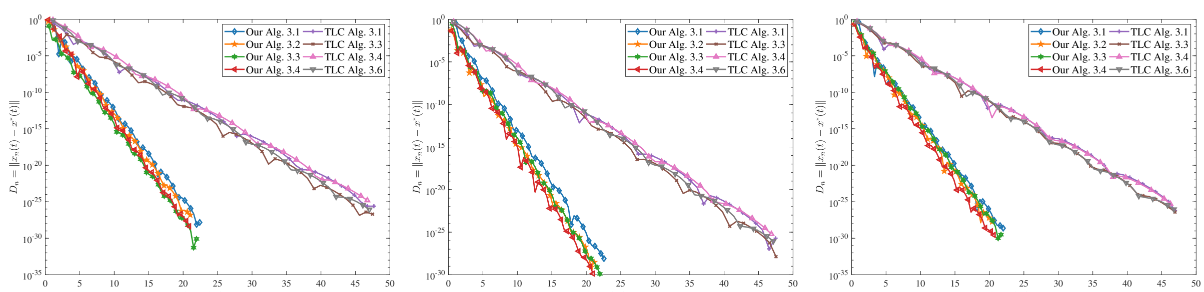
and the operator $Q : \mathscr{H} \to \mathscr{H}$ be given by

$$Qb = (R - \|b\|)b, \quad \forall b \in \mathscr{H}.$$

TABLE 1. Numerical results of all algorithms with different dimensions for Example 4.1

Algorithms	$m=20$		$m=50$		$m=100$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	3.60E-13	0.0350	3.12E-09	0.0314	1.13E-07	0.0335
Our Alg. 3.2	9.23E-12	0.0221	3.71E-08	0.0208	2.03E-07	0.0269
Our Alg. 3.3	1.86E-14	0.0222	1.06E-07	0.0238	1.26E-06	0.0331
Our Alg. 3.4	9.48E-12	0.0246	6.05E-07	0.0204	3.06E-06	0.0271
TLC Alg. 3.1	1.03E-08	0.0618	3.06E-07	0.0468	1.76E-06	0.0940
TLC Alg. 3.3	2.52E-09	0.0526	2.01E-07	0.0429	3.91E-06	0.0825
TLC Alg. 3.4	1.03E-08	0.0437	3.06E-07	0.0454	1.76E-06	0.0626
TLC Alg. 3.6	2.52E-09	0.0398	2.01E-07	0.0416	3.91E-06	0.0609

The solution of the variational inequality problem (VIP) with Q and $\mathscr{X}$ given above is $b^*(t)=0$. It is not hard to check that operator Q is pseudomonotone rather than monotone, and thus the algorithms proposed in the literature (see, e.g., [17, 18, 19, 20]) for solving monotone VIPs will not be available. For this experiment, we choose $R=1.5$, $r=1$, $k=1.1$. The parameters of all algorithms are set as follows. Select $\phi=0.2$, $\varepsilon_n=1/(n+1)^2$, $\eta=1.5$, $\chi_n=1/(n+1)$, $\tau_n=0.9(1-\chi_n)$, and $f(b)=0.1b$ for all algorithms. Adopt $\beta=1.0$, $\lambda_1=0.1$, $\mu=0.4$, $\alpha_n=1+1/n$, $\xi_n=1+1/(n+1)^{1.1}$, and $\rho_n=1/(n+1)^{1.1}$ for the proposed algorithms. Choose $\delta=2$, $\zeta=0.5$, $\phi=0.4$ for TLC Alg. 3.1, TLC Alg. 3.3, TLC Alg. 3.4 and TLC Alg. 3.6. The maximum number of iterations 50 is used as a common stopping criterion. We employ $D_n=\|b_n(t)-b^*(t)\|$ as the error measure for the n-th step of all algorithms. The performance of all methods with three different initial values are stated in Fig. 5 and Table 2.



(a) $b_0(t)=b_1(t)=10t^3$ (b) $b_0(t)=b_1(t)=\mathrm{e}^t$ (c) $b_0(t)=b_1(t)=\log(t)$

FIGURE 5. The behavior of all algorithms with different initial values for Example 4.2

As shown in Fig. 5 and Table 2, the suggested approaches outperform the methods incorporated with Armijo-type step size introduced by Tan et al. [25] in terms of accuracy and convergence, and this discovery is unrelated to the choice of initial values. It is worth emphasizing that our approaches employ an adaptive non-monotonic step size rule, which enables them to perform iterative step size updates using a simple computation based on previously known information. However, the algorithms with Armijo-type step size in [25] take more time in infinite-dimensional spaces because they may need to calculate the projection on $\mathscr{X}$ multiple times for each iteration

TABLE 2. Numerical results of all algorithms with different initial values for Example 4.2

Algorithms	$b_0(t) = b_1(t) = 10t^3$		$b_0(t) = b_1(t) = \mathrm{e}^t$		$b_0(t) = b_1(t) = \log(t)$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	1.49E-28	22.4082	7.91E-29	22.5626	2.39E-29	21.9969
Our Alg. 3.2	1.58E-27	21.0311	2.94E-29	21.1421	1.82E-28	20.3942
Our Alg. 3.3	8.12E-31	21.9393	1.26E-30	21.9759	3.11E-30	21.6810
Our Alg. 3.4	4.32E-29	20.8793	1.37E-30	20.9848	2.86E-30	20.6285
TLC Alg. 3.1	2.44E-26	47.6775	1.94E-26	47.4733	6.06E-27	46.9403
TLC Alg. 3.3	1.99E-27	47.4819	1.32E-28	47.5316	3.79E-27	46.9302
TLC Alg. 3.4	1.55E-25	46.7254	6.04E-26	46.8658	4.99E-26	46.2634
TLC Alg. 3.6	8.69E-27	47.0215	8.75E-27	47.1401	9.09E-27	46.7680

to determine the suitable step size. Therefore, the non-monotonic step size criterion introduced in this paper is preferable to the Armijo-type step size criterion.

5. CONCLUSIONS

This paper introduces four modified adaptive numerical approaches for discovering the solutions to pseudomonotone VIPs in infinite-dimensional Hilbert spaces. The suggested methods use a novel adaptive step size criteria that do not need a line search, allowing them to work with the loss of previous knowledge of the Lipschitz constant of the mapping involved. Under certain moderate restrictions put on the parameters, strong convergence theorems for the suggested approaches are confirmed. Finally, two numerical examples are provided to confirm the advantages of our proposed methods with respect to previously known ones. How to perform the convergence analysis of the proposed algorithms and the extension of our algorithms to Banach spaces are future research directions.

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