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GLOBAL AND LINEAR CONVERGENCE OF ALTERNATED INERTIAL SINGLE PROJECTION ALGORITHMS FOR PSEUDO-MONOTONE VARIATIONAL INEQUALITIES

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Abstract. In this paper, we investigate three new relaxed single projection methods with alternating inertial extrapolation steps and adaptive non-monotonic step sizes for solving pseudo-monotone variational inequalities in real Hilbert spaces. The proposed algorithms need to compute the projection on the feasible set only once in each iteration and they can work adaptively without the prior information of the Lipschitz constant of the mapping. The weak convergence theorems of the proposed iterative schemes are established under some appropriate conditions imposed on the parameters. These methods recover the Fejér monotonicity of the even subsequence with respect to the solution and obtain linear convergence rates. Finally, some numerical experiments and applications to optimal control problems are provided to demonstrate the advantages and efficiency of the proposed methods compared to some recent related ones.
Key Words and Phrases: Variational inequality, alternated inertial method, projection and contraction method, subgradient extragradient method, pseudo-monotone operator, adaptive stepsize.
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1. Introduction

Our goal in this paper is to construct several fast iterative algorithms to solve variational inequality problems in the framework of real Hilbert spaces. Let C be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and $A:\mathcal{H}\to\mathcal{H}$ be a nonlinear operator. The variational inequality problem for A on C is described as follows:

$$\text{find } x^*\in C \text{ such that } \langle Ax^*, x-x^*\rangle\geq 0,\quad \forall x\in C. \tag{VIP}$$

The solution set of (VIP) is denoted as $\mathrm{VI}(C, A)$. The variational inequality model plays an important role in many fields and it constructs a simple framework for many optimization problems. Some recent applications of variational inequalities can be found in [2, 4, 9, 10] and the references therein.

The simplest way to solve (VIP) is the gradient-projection algorithm, which involves computing only one projection on the feasible set in each iteration. However, the drawback of this method is that its convergence requires the operator to be strongly monotonic, which is a slightly strong and restrictive assumption. To overcome this difficulty, Korpelevich [23] proposed a two-step iterative scheme known as the extragradient method (shortly, EGM), which includes calculating the projection on the feasible set twice in each iteration. The convergence of the EGM was proved under the condition that the operator is monotone (or even, pseudo-monotone). It is known that computing the projection is not easy, especially when the form of the feasible set is complex. Recently, some methods that require calculating only one projection on the feasible set were proposed to solve (VIP); such as the projection and contraction method [17], the Tseng extragradient method [43], the subgradient extragradient method [5, 6, 7], and the projected gradient method [27, 26]. These methods greatly improve the computational efficiency of the EGM due to the fact that they reduce the computation of one projection on the feasible set in each iteration. In recent years, a large number of variants and improved forms were proposed by many scholars based on the methods; see, e.g., [13, 16, 37, 34, 41] and the references therein.

Recently, the inertial extrapolation method, which is based on a discrete version of a second-order dissipative dynamical system (see [1, 32]), was widely studied by scholars as one of the techniques to accelerate the convergence speed of algorithms. The main idea of inertial-type methods is that the next iteration depends on the combination of the previous two (or more) iterations. Noting this small change can improve the convergence speed of the methods used. Some inertial projection-based algorithms for solving variational inequality problems can be found in [8, 14, 12, 15, 19, 39, 40] and the references therein. Very recently, Shehu et al. [40] proposed two new inertial projection-type algorithms, which are based on the inertial method, the projection and contraction method, and the relaxation method, for solving variational inequality problems. Specifically, their algorithms are stated in Algorithms 1.1 and 1.2 below.

We observe that the Algorithm 1.1 and the Algorithm 1.2 compute x_{n+1} differently in the third step, which results in different ranges of values for their inertial parameter α and relaxation parameter θ. Indeed, the Algorithm 1.1 is an over-relaxed projection and contraction method (see [40, Remark 3.5]), while the Algorithm 1.2 is an under-relaxed version. In addition, the upper bound of the inertial parameter α in Dong et al.'s Algorithm 3.1 [12] and Shehu et al.'s Algorithm 1.1 [40] is less than 1, while this upper bound is extended to equal 1 in the Algorithm 1.2 introduced in [40] (i.e., $\alpha = 1$ is available for Algorithm 1.2). Moreover, Shehu et al. [40] established the linear convergence of the iterative sequence generated by the Algorithm 1.2. Some numerical experiments are also provided to demonstrate the computational efficiency of the proposed algorithms with respect to some related schemes.

On the other hand, it is worth noting that non-inertial projection methods enjoy the Fejér monotonicity of the iterative sequence with respect to the solution, which is

Algorithm 1.1 The Algorithm 3.2 of Shehu et al. [40]

Initialization: Take $\gamma \in (0,2)$, $\alpha \in [0,1)$, $\mu \in (0,1)$, $\lambda_1 > 0$, and

$$\theta \in \left(0, \frac{2(1-\alpha)^2}{\gamma\alpha(1+\alpha)+\gamma(1-\alpha)^2}\right).$$

Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:

Step 1. Compute $w_n = x_n + \alpha\,(x_n - x_{n-1})$.

Step 2. Compute $y_n = P_C\,(w_n - \lambda_n A w_n)$, where

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\,\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \lambda_n\right\}, & Aw_n \neq Ay_n; \\ \lambda_n, & \text{otherwise}. \end{cases} \tag{S1}$$

If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = (1-\theta)w_n + \theta\,(w_n - \gamma\eta_n d_n)$, where

$$\eta_n = \begin{cases} \dfrac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0; \\ 0, & d_n = 0, \end{cases} \quad \text{and} \quad d_n = w_n - y_n - \lambda_n\,(Aw_n - Ay_n)\,. \tag{1.1}$$

Set $n := n+1$ and go to **Step 1**.

Algorithm 1.2 The Algorithm 4.1 of Shehu et al. [40]

Initialization: Take $\gamma \in (0,2)$, $\alpha \in [0,1]$, $\mu \in (0,1)$, $\lambda_1 > 0$, and $\theta \in (0,1/2)$.

Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:

Step 1. Compute $w_n = x_n + \alpha\,(x_n - x_{n-1})$.

Step 2. Compute $y_n = P_C\,(w_n - \lambda_n A w_n)$, and update λ_{n+1} by (S1). If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = (1-\theta)x_n + \theta\,(w_n - \gamma\eta_n d_n)$, where η_n and d_n are defined in (1.1).

Set $n := n+1$ and go to **Step 1**.

not being enjoyed by their corresponding inertial versions for variational inequalities. This lack of Fejér monotonicity causes inertial projection methods for variational inequalities sometimes not to converge faster than their corresponding non-inertial versions. Recently, Mu and Peng [28] proposed an alternated inertial method that recovers the Fejér monotonicity of even subsequences to overcome the above problem. This alternated inertial idea was further extended by many authors to some iterative algorithms for solving variational inequalities, fixed point problems, split feasibilities, split equality problems, split common null point problems, and other optimization problems; see, e.g., [20, 21, 22, 11, 29, 38, 36, 35] and the references therein. Their numerical experiments show that alternated inertial methods exhibit better performance than inertial-type ones.

It is our aim in this paper to present several alternated inertial projection methods to solve the (VIP). We next review several alternated inertial methods that already exist in the literature [29, 38] for solving variational inequality problems, which motivate us to explore some new iterative schemes. Recently, based on the alternated inertial method [28] and the projection and contraction method [17], Shehu and Iyiola [38] proposed two alternated inertial projection algorithms (one with a fixed step size and the other with an adaptive step size) for solving pseudo-monotone variational inequalities in real Hilbert spaces. More precisely, the adaptive scheme suggested by Shehu and Iyiola [38] is described in Algorithm 1.3 below.

Algorithm 1.3 The Algorithm 2 of Shehu and Iyiola [38]

Initialization: Take $\gamma \in (0,2)$, $0 \leq \alpha_n \leq \alpha < (2-\gamma)/\gamma$, $\mu \in (0,1)$, and $\lambda_1 > 0$.
Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:
Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}; \\ x_n + \alpha_n \left(x_n - x_{n-1}\right), & n = \text{odd}. \end{cases} \tag{1.2}$$

Step 2. Compute $y_n = P_C\left(w_n - \lambda_n A w_n\right)$, and update λ_{n+1} by (S1). If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = w_n - \gamma \eta_n d_n$, where η_n and d_n are defined in (1.1).
Set $n := n+1$ and go to **Step 1**.

Under some suitable conditions, Shehu and Iyiola [38] established a weak convergence theorem for the proposed Algorithm 1.3. Under the assumption that the operator A is strongly pseudo-monotone, they also verified the R-linear convergence rate of a new scheme (i.e., the scheme resulting from missing step 3 in Algorithm 1.3 and changing y_n to x_{n+1} in the second step). The computational efficiency of the algorithms proposed by Shehu and Iyiola [38] compared to some known inertial projection and contraction methods is illustrated by several numerical examples in finite-dimensional spaces.

Very recently, inspired by the Tseng extragradient method [43] and the work of Shehu and Iyiola [38], Ogbuisi, Shehu and Yao [29] introduced a new alternated inertial Tseng exragradient method with relaxation effects and adaptive step sizes to discover solutions of pseudo-monotone inequality problems in real Hilbert spaces. Indeed, their scheme is shown in Algorithm 1.4 below.

Algorithm 1.4 The Algorithm 3.1 of Ogbuisi, Shehu and Yao [29]

Initialization: Take $\theta \in (0,1]$, $0 \leq \alpha_n \leq (1-\mu)^2/(1+\mu)^2$, $\mu \in (0,1)$, and $\lambda_1 > 0$.
Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:
Step 1. Compute w_n by (1.2).
Step 2. Compute $y_n = P_C\left(w_n - \lambda_n A w_n\right)$, and update λ_{n+1} by (S1). If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = (1-\theta) w_n + \theta z_n$, where $z_n = y_n - \lambda_n \left(A y_n - A w_n\right)$.
Set $n := n+1$ and go to **Step 1**.

Under some mild conditions, Ogbuisi et al. [29] confirmed that the iterative sequence generated by the proposed Algorithm 1.4 converges weakly to the solution of (VIP) and further obtained the R-linear convergence rate of Algorithm 1.4. They provide some numerical examples occurring in a finite-dimensional space to demonstrate the computational performance of the proposed algorithm in comparison with some relaxed inertial projection-type methods. Notice that the Algorithms 1.3 and 1.4 need to compute the projection on the feasible set only once in each iteration and can recover the Fejér monotonicity of the even subsequence with respect to the solution. These two advantages make their performance better than some known inertial projection-type algorithms. On the other hand, it is worth noting that the adaptive step size criterion (S1) generates a non-increasing sequence of steps, which further may affect the computational efficiency of the algorithms used.

Inspired and motivated by the above work, in this paper, we propose three new alternated inertial extragradient methods with adaptive non-monotonic step sizes and relaxation effects to find solutions of pseudo-monotone variational inequality problems. Our algorithms need to perform the projection computation on the feasible set only once in each iteration, and they also recover the Fejér monotonicity of the even subsequence with respect to the solution. Under some mild and suitable conditions, the weak convergence theorems of the proposed methods are established in real Hilbert spaces. Moreover, the R-linear convergence rates of the proposed algorithms are verified under the condition that the operators are strongly pseudo-monotone. Finally, some numerical tests occurring in finite- and infinite-dimensional spaces and applications to optimal control problems are given to illustrate the advantages and efficiency of the proposed algorithms over some known inertial iterative schemes.

The outline of this paper is as follows. In Section 2, we review some definitions and lemmas that need to be used in the sequel. Section 3 is devoted to stating three alternated inertial algorithms and analyzing their convergence. The linear convergence rates of the proposed algorithms under the condition that the operators are strongly pseudo-monotone are given in Section 4. Section 5 illustrates the computational performance of the proposed algorithms with respect to some known iterative methods through several numerical experiments and applications. Finally, the paper is concluded by a brief summary in Section 6, the last section.

2. Preliminaries

Let C be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{R}$, we have the following inequality.

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2. \tag{2.1}$$

Definition 2.1. Let C be a nonempty, closed and convex subset of $\mathcal{H}$. P_C is called the metric projection of $\mathcal{H}$ onto C if, for any point $x \in \mathcal{H}$, there exists a unique point $P_C(x) \in C$ such that $\|x - P_C(x)\| \le \|x - y\|$, $\forall y \in C$.

It is known that P_C has the following basic properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \le 0, \quad \forall x \in \mathcal{H}, \forall y \in C, \tag{2.2}$$

and

$$\|P_C(x) - y\|^2 \le \|x-y\|^2 - \|x - P_C(x)\|^2, \quad \forall x \in \mathcal{H}, y \in C. \tag{2.3}$$

Definition 2.2. A mapping $A : \mathcal{H} \to \mathcal{H}$ is said to be

(1) *η-strongly monotone* on $\mathcal{H}$ if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \eta\|x - y\|^2,\ \forall x, y \in \mathcal{H};$$

(2) *monotone* if $\langle Ax - Ay, x - y\rangle \geq 0,\ \forall x, y \in \mathcal{H}$.

(3) *δ-strongly pseudo-monotone* on $\mathcal{H}$ if there exists $\delta > 0$ such that

$$\langle Ay, x - y\rangle \geq 0 \Rightarrow \langle Ax, x - y\rangle \geq \delta\|x - y\|^2,\ \forall x, y \in \mathcal{H};$$

(4) *pseudo-monotone* if $\langle Ax, y - x\rangle \geq 0 \Rightarrow \langle Ay, y - x\rangle \geq 0,\ \forall x, y \in \mathcal{H}$.

(5) *L-Lipschitz continuous* with $L > 0$ if $\|Ax - Ay\| \leq L\|x - y\|,\ \forall x, y \in \mathcal{H}$.

(6) *sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies that $\{Ax_n\}$ converges weakly to Ax.

Remark 2.1. Note that (1) implies (2), (1) implies (3), (3) implies (4), and (2) implies (4) in the above definitions. Furthermore, if (3) is satisfied, then the (VIP) has a unique solution.

Definition 2.3. A sequence $\{x_n\}$ in $\mathcal{H}$ is said to be

- *Fejér monotone* with respect to a set Q if each point in the sequence is not strictly farther from any point in Q than its predecessor. That is,

$$\|x_{n+1} - x\| \leq \|x_n - x\|, \forall x \in Q;$$

- *convergent weakly* to $p \in \mathcal{H}$ if $\lim_{n\to\infty} \langle x_n, x\rangle = \langle p, x\rangle$, $\forall x \in \mathcal{H}$;
- *convergent R-linearly* to x^* with rate $\eta \in [0, 1)$ if there exists a constant $c > 0$ such that $\|x_n - x^*\| \leq c\eta^n$, $\forall n \in \mathbb{N}$.

Lemma 2.1 ([31])**.** *Let $\{\lambda_n\}$, $\{\xi_n\}$, and $\{\zeta_n\}$ be three sequences of nonnegative numbers such that*

$$\lambda_{n+1} \leq \xi_n\lambda_n + \zeta_n,\ \forall n \in \mathbb{N}.$$

If $\{\xi_n\} \subset [1, +\infty)$, $\sum_{n=1}^{\infty}(\xi_n - 1) < \infty$, and $\sum_{n=1}^{\infty}\zeta_n < \infty$, then $\lim_{n\to\infty}\lambda_n$ exists.

Lemma 2.2 ([30])**.** *Let C be a nonempty set of $\mathcal{H}$, and $\{x_n\}$ be a sequence in $\mathcal{H}$. If $\lim_{n\to\infty}\|x_n - x\|$ exists for any $x \in C$, and every sequential weak cluster point of $\{x_n\}$ is in C, then $\{x_n\}$ converges weakly to a point in C.*

3. Main Results

In this section, we present three new alternated inertial single projection methods with adaptive non-monotonic step sizes and relaxation effects to discover solutions of pseudo-monotone variational inequalities in real Hilbert spaces. We first assume that the following conditions hold in order to analyze the convergence of the proposed algorithms.

(C1) The feasible set C is a nonempty, closed, and convex subset of $\mathcal{H}$.

(C2) The solution set of the (VIP) is nonempty, that is, $\mathrm{VI}(C, A) \neq \emptyset$.

(C3) The mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudo-monotone, L-Lipschitz continuous and sequentially weakly continuous on bounded subsets of $\mathcal{H}$.

3.1. **The first algorithm.** In this subsection, we introduce a relaxed and modified subgradient extragradient method with alternating inertial extrapolation steps to solve the variational inequality problem (VIP). Now, our first proposed method is shown in Algorithm 3.1 below.

Algorithm 3.1 The first modified subgradient extragradient method for (VIP).

Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:
Step 1. Compute

$$w_n = \begin{cases} x_n, & n = \text{even}; \\ x_n + \alpha_n (x_n - x_{n-1}), & n = \text{odd}. \end{cases} \tag{3.1}$$

Step 2. Compute $y_n = P_C (w_n - \lambda_n A w_n)$, where

$$\lambda_{n+1} = \begin{cases} \min \left\{ \dfrac{\mu q_n \|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \xi_n \lambda_n + \zeta_n \right\}, & Aw_n \neq Ay_n; \\ \xi_n \lambda_n + \zeta_n, & \text{otherwise}. \end{cases} \tag{S2}$$

If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = (1-\theta) w_n + \theta z_n$, where

$$z_n = P_{T_n}(w_n - \beta \lambda_n A y_n),$$

and

$$T_n = \{x \in \mathcal{H} \mid \langle w_n - \lambda_n A w_n - y_n, x - y_n \rangle \leq 0\}. \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1. Notice that the proposed Algorithm 3.1 and the subgradient extragradient method suggested by Censor et al. [5, 6, 7] are different in computing z_n. Specifically, in updating z_n we utilize the projection from $w_n - \beta \lambda_n A y_n$ to T_n, while Censor et al. uses the projection from $w_n - \lambda_n A y_n$ to T_n. We insert a new parameter β and the computational advantage of the proposed Algorithm 3.1 is illustrated by the numerical experiments given in Sect. 5. Furthermore, if the parameters $\theta = 1$ and $\beta = 1$ in Algorithm 3.1, then it turns out to be an alternated inertial version of the subgradient extragradient method (see [5, 6, 7]) for solving variational inequalities. It is noted that the proposed Algorithm 3.1 can solve pseudo-monotone variational inequalities, while the methods proposed in [5, 6, 7] can only solve monotone variational inequalities. Therefore, the proposed Algorithm 3.1 has a wide range of applications.

To perform the convergence analysis of the proposed Algorithm 3.1, we assume that it satisfies the following conditions (C4) and (C5).

(C4) Let $\lambda_1 > 0$, $\mu \in (0,1)$, $\{q_n\} \subset [1, \infty)$ such that $\lim_{n \to \infty} q_n = 1$, $\{\xi_n\} \subset [1, \infty)$ such that $\sum_{n=0}^{\infty} (\xi_n - 1) < \infty$, and $\{\zeta_n\} \subset [0, \infty)$ such that $\sum_{n=0}^{\infty} \zeta_n < \infty$.

(C5) Let $\theta \in (0,1]$, $\beta \in (0, 2/(1+\mu))$, $0 \leq \alpha_n \leq \alpha < \dfrac{\beta^* + 2(1-\theta)}{2\theta}$, where $\beta^* = 2 - \beta - \beta\mu$ when $\beta \in [1, 2/(1+\mu))$ and $\beta^* = \beta - \beta\mu$ when $\beta \in (0,1)$.

Remark 3.2. It is easy to verify that Condition (C4) is easily satisfied, for example, by taking

$$q_n = (n+1)/n,\ \xi_n = 1 + 1/(n+1)^{1.1} \text{ and } \zeta_n = 1/(n+1)^{1.1}.$$

In addition, note that the inertial parameter α_n in Condition (C5) is allowed to be greater than or equal to 1 when the relaxation parameter $\theta \in (0,1)$, for instance, by choosing $\theta = 0.5$, then $\alpha_n \le \alpha < \beta^* + 1$ ($\beta^* > 0$ for all $\beta \in (0, 2/(1+\mu))$).

The following lemmas are important for the convergence analysis of our main results.

Lemma 3.1. *Suppose that Conditions (C2) and (C4) hold. Then the sequence* $\{\lambda_n\}$ *generated by* (S2) *is well defined and* $\lim_{n\to\infty} \lambda_n$ *exists.*

Proof. Since A is Lipschitz continuous with $L > 0$ and $q_n \ge 1$, one sees that

$$\frac{\mu q_n \|w_n - y_n\|}{\|Aw_n - Ay_n\|} \ge \frac{\mu q_n \|w_n - y_n\|}{L\|w_n - y_n\|} \ge \frac{\mu}{L}.$$

Thus

$$\lambda_{n+1} = \min\left\{ \frac{\mu q_n \|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \xi_n \lambda_n + \zeta_n \right\} \ge \min\left\{ \frac{\mu}{L}, \lambda_n \right\},$$

where $\xi_n \ge 1$ and $\zeta_n > 0$. By induction, one obtains that the sequence $\{\lambda_n\}$ has a lower bound $\{\mu/L, \lambda_1\}$. It follows from (S2) that

$$\lambda_{n+1} \le \xi_n \lambda_n + \zeta_n.$$

Thanks to Condition (C4) holds, which together with Lemma 2.1 produces that $\lim_{n\to\infty} \lambda_n$ exists. That is the desired result. □

Remark 3.3. It should be noted that the step size update criterion (S2) is preferable to (S1) due to the fact that the step size sequence generated by (S2) is non-monotonic (i.e., it allows $\lambda_{n+1} \ge \lambda_n$ for some n), whereas the step size update criterion (S1) produces a non-increasing step size sequence (i.e., it must satisfy $\lambda_{n+1} \le \lambda_n$ for all $n \ge 1$). Furthermore, our step size criterion (S2) contains some known step size update methods in the literature. To see this, we next enumerate some special cases of (S2). If $q_n = \xi_n = 1$ and $\zeta_n = 0$ in (S2), then it reduces to (S1), which is used by many authors (e.g., [29, 38]). If $q_n = \xi_n = 1$ and $\zeta_n \ne 0$ in (S2), then it evolves to the step size proposed by Liu and Yang [24]. If $q_n = 1$ in (S2), then it degenerates to the step size recently introduced by Ma and Liu [25].

Lemma 3.2. *Assume that Conditions (C2) and (C5) hold, and the sequence* $\{x_n\}$ *is generated by Algorithm 3.1. Then the sequence* $\{x_{2n}\}$ *is Fejér monotone with respect to* $\mathrm{VI}(C,A)$ *and* $\lim_{n\to\infty} \|x_{2n} - x^*\|$ *exists, where* $x^* \in \mathrm{VI}(C,A)$*. Furthermore,*

$$\lim_{n\to\infty} \|x_{2n} - y_{2n}\| = 0$$

and

$$\lim_{n\to\infty} \|x_{2n+1} - x_{2n}\| = 0.$$

Proof. By the definition of z_{2n+1} and (2.3), we obtain

$$\begin{aligned}
&\|z_{2n+1}-x^*\|^2\\
=&\left\|P_{T_{2n+1}}\left(w_{2n+1}-\beta\lambda_{2n+1}Ay_{2n+1}\right)-x^*\right\|^2\\
\le&\|w_{2n+1}-\beta\lambda_{2n+1}Ay_{2n+1}-x^*\|^2-\|w_{2n+1}-\beta\lambda_{2n+1}Ay_{2n+1}-z_{2n+1}\|^2\\
=&\|w_{2n+1}-x^*\|^2+(\beta\lambda_{2n+1})^2\|Ay_{2n+1}\|^2\\
&-2\langle w_{2n+1}-x^*,\beta\lambda_{2n+1}Ay_{2n+1}\rangle-\|w_{2n+1}-z_{2n+1}\|^2\\
&-(\beta\lambda_{2n+1})^2\|Ay_{2n+1}\|^2+2\langle w_{2n+1}-z_{2n+1},\beta\lambda_{2n+1}Ay_{2n+1}\rangle\\
=&\|w_{2n+1}-x^*\|^2-\|w_{2n+1}-z_{2n+1}\|^2-2\langle\beta\lambda_{2n+1}Ay_{2n+1},z_{2n+1}-x^*\rangle\\
=&\|w_{2n+1}-x^*\|^2-\|w_{2n+1}-z_{2n+1}\|^2-2\langle\beta\lambda_{2n+1}Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle\\
&-2\langle\beta\lambda_{2n+1}Ay_{2n+1},y_{2n+1}-x^*\rangle.
\end{aligned}\tag{3.3}$$

Since $x^*\in \mathrm{VI}(C,A)$ and $y_{2n+1}\in C$, we have $\langle Ax^*,y_{2n+1}-x^*\rangle\ge 0$. This together with the pseudo-monotonicity of mapping A implies that $\langle Ay_{2n+1},y_{2n+1}-x^*\rangle\ge 0$. Therefore, (3.3) reduces to

$$\begin{aligned}
\|z_{2n+1}-x^*\|^2\le&\|w_{2n+1}-x^*\|^2-\|w_{2n+1}-z_{2n+1}\|^2\\
&-2\langle\beta\lambda_{2n+1}Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle.
\end{aligned}\tag{3.4}$$

Now we estimate $2\langle\beta\lambda_{2n+1}Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle$. Note that

$$\begin{aligned}
-\|w_{2n+1}-z_{2n+1}\|^2=&-\|w_{2n+1}-y_{2n+1}\|^2-\|y_{2n+1}-z_{2n+1}\|^2\\
&+2\langle w_{2n+1}-y_{2n+1},z_{2n+1}-y_{2n+1}\rangle.
\end{aligned}\tag{3.5}$$

In addition,

$$\begin{aligned}
&\langle w_{2n+1}-y_{2n+1},z_{2n+1}-y_{2n+1}\rangle\\
=&\langle w_{2n+1}-y_{2n+1}-\lambda_{2n+1}Aw_{2n+1}+\lambda_{2n+1}Aw_{2n+1}\\
&-\lambda_{2n+1}Ay_{2n+1}+\lambda_{2n+1}Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle\\
=&\langle w_{2n+1}-\lambda_{2n+1}Aw_{2n+1}-y_{2n+1},z_{2n+1}-y_{2n+1}\rangle\\
&+\lambda_{2n+1}\langle Aw_{2n+1}-Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle\\
&+\langle\lambda_{2n+1}Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle.
\end{aligned}\tag{3.6}$$

Since $z_{2n+1}\in T_{2n+1}$, one has

$$\langle w_{2n+1}-\lambda_{2n+1}Aw_{2n+1}-y_{2n+1},z_{2n+1}-y_{2n+1}\rangle\le 0.\tag{3.7}$$

According to the definition of λ_{2n+1}, it is easy to obtain

$$\begin{aligned}
&\langle Aw_{2n+1}-Ay_{2n+1},z_{2n+1}-y_{2n+1}\rangle\\
\le&\|Aw_{2n+1}-Ay_{2n+1}\|\,\|z_{2n+1}-y_{2n+1}|\\
\le&\frac{\mu q_{2n+1}}{\lambda_{2n+2}}\|w_{2n+1}-y_{2n+1}\|\,\|z_{2n+1}-y_{2n+1}|\\
\le&\frac{\mu q_{2n+1}}{2\lambda_{2n+2}}\left(\|w_{2n+1}-y_{2n+1}\|^2+\|z_{2n+1}-y_{2n+1}\|^2\right).
\end{aligned}\tag{3.8}$$

Let

$$a_{2n+1} := \|w_{2n+1} - y_{2n+1}\|^2 + \|z_{2n+1} - y_{2n+1}\|^2 . \tag{3.9}$$

Substituting (3.6), (3.7), (3.8) and (3.9) into (3.5), we obtain

$$\begin{aligned} - \|w_{2n+1} - z_{2n+1}\|^2 \leq\ & 2 \langle \lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ & - \left(1 - \frac{\mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}}\right) a_{2n+1}, \end{aligned} \tag{3.10}$$

From (3.10) (noting that $\beta > 0$), one has

$$\begin{aligned} & - 2 \langle \beta \lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ & \leq -\beta \left(1 - \frac{\mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}}\right) a_{2n+1} + \beta \|w_{2n+1} - z_{2n+1}\|^2 . \end{aligned} \tag{3.11}$$

Combining (3.4) and (3.11), we have

$$\begin{aligned} \|z_{2n+1} - x^*\|^2 \leq\ & \|w_{2n+1} - x^*\|^2 - (1 - \beta) \|w_{2n+1} - z_{2n+1}\|^2 \\ & - \beta \left(1 - \frac{\mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}}\right) a_{2n+1}. \end{aligned} \tag{3.12}$$

Note that

$$\|w_{2n+1} - z_{2n+1}\|^2 \leq 2 \left(\|w_{2n+1} - y_{2n+1}\|^2 + \|z_{2n+1} - y_{2n+1}\|^2\right) = 2a_{2n+1},$$

which yields that

$$-(1 - \beta) \|w_{2n+1} - z_{2n+1}\|^2 \leq -2(1 - \beta) a_{2n+1}, \ \forall \beta \geq 1.$$

This together with (3.12) implies

$$\|z_{2n+1} - x^*\|^2 \leq \|w_{2n+1} - x^*\|^2 - \left(2 - \beta - \frac{\beta \mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}}\right) a_{2n+1}, \ \forall \beta \geq 1.$$

On the other hand, if $\beta \in (0, 1)$, then we obtain

$$\|z_{2n+1} - x^*\|^2 \leq \|w_{2n+1} - x^*\|^2 - \beta \left(1 - \frac{\mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}}\right) a_{2n+1}, \ \forall \beta \in (0, 1).$$

Therefore, we conclude that

$$\|z_{2n+1} - x^*\|^2 \leq \|w_{2n+1} - x^*\|^2 - \beta^*_{2n+1} a_{2n+1}, \tag{3.13}$$

where

$$\beta^*_{2n+1} = 2 - \beta - \frac{\beta \mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}} \text{ when } \beta \in [1, 2/(1 + \mu))$$

and

$$\beta^*_{2n+1} = \beta - \frac{\beta \mu q_{2n+1} \lambda_{2n+1}}{\lambda_{2n+2}} \text{ when } \beta \in (0, 1).$$

By Condition (C4) and Lemma 3.1, we have

$$\beta^* := \lim_{n \to \infty} \beta^*_{2n+1} = \begin{cases} 2 - \beta - \beta\mu, & \beta \in [1, 2/(1 + \mu)); \\ \beta - \beta\mu, & \beta \in (0, 1). \end{cases}$$

Thus, $\lim_{n \to \infty} \beta^*_{2n+1} > 0$ for all $\beta \in (0, 2/(1 + \mu))$. There exists a positive constant N_0 such that $\beta^*_{2n+1} > 0$ holds for all $n \geq N_0$.

From (2.1) and (3.13) (noting that $\|w_{2n+1}-z_{2n+1}\|^2 \leq 2a_{2n+1}$), we obtain

$$\begin{aligned}
\|x_{2n+2}-x^*\|^2 &= \|(1-\theta)(w_{2n+1}-x^*)+\theta(z_{2n+1}-x^*)\|^2 \\
&= (1-\theta)\|w_{2n+1}-x^*\|^2+\theta\|z_{2n+1}-x^*\|^2 \\
&\quad -\theta(1-\theta)\|w_{2n+1}-z_{2n+1}\|^2 \\
&\leq (1-\theta)\|w_{2n+1}-x^*\|^2+\theta\|w_{2n+1}-x^*\|^2 \\
&\quad -\theta\beta^*_{2n+1}a_{2n+1}-\theta(1-\theta)\|w_{2n+1}-z_{2n+1}\|^2 \\
&= \|w_{2n+1}-x^*\|^2-\theta\beta^*_{2n+1}a_{2n+1}-\theta(1-\theta)\|w_{2n+1}-z_{2n+1}\|^2 \\
&\leq \|w_{2n+1}-x^*\|^2-\theta\left(\frac{1}{2}\beta^*_{2n+1}+(1-\theta)\right)\|w_{2n+1}-z_{2n+1}\|^2 .
\end{aligned} \tag{3.14}$$

By (3.14) (noting that $w_{2n}=x_{2n}$), we observe that

$$\begin{aligned}
\|x_{2n+1}-x^*\|^2 &\leq \|w_{2n}-x^*\|^2-\theta\left(\frac{1}{2}\beta^*_{2n}+(1-\theta)\right)\|w_{2n}-z_{2n}\|^2 \\
&= \|x_{2n}-x^*\|^2-\theta\left(\frac{1}{2}\beta^*_{2n}+(1-\theta)\right)\|x_{2n}-z_{2n}\|^2 .
\end{aligned} \tag{3.15}$$

Combining (2.1) and (3.15), we have

$$\begin{aligned}
&\|w_{2n+1}-x^*\|^2 \\
&= \|x_{2n+1}+\alpha_{2n+1}(x_{2n+1}-x_{2n})-x^*\|^2 \\
&= \|(1+\alpha_{2n+1})(x_{2n+1}-x^*)-\alpha_{2n+1}(x_{2n}-x^*)\|^2 \\
&= (1+\alpha_{2n+1})\|x_{2n+1}-x^*\|^2-\alpha_{2n+1}\|x_{2n}-x^*\|^2 \\
&\quad +\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2 \\
&\leq (1+\alpha_{2n+1})\left[\|x_{2n}-x^*\|^2-\theta\left(\frac{1}{2}\beta^*_{2n}+(1-\theta)\right)\|x_{2n}-z_{2n}\|^2\right] \\
&\quad -\alpha_{2n+1}\|x_{2n}-x^*\|^2+\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2 \\
&= \|x_{2n}-x^*\|^2-\theta(1+\alpha_{2n+1})\left(\frac{1}{2}\beta^*_{2n}+(1-\theta)\right)\|x_{2n}-z_{2n}\|^2 \\
&\quad +\alpha_{2n+1}(1+\alpha_{2n+1})\|x_{2n+1}-x_{2n}\|^2 .
\end{aligned} \tag{3.16}$$

By the definition of x_{2n+1} (noting that $w_{2n}=x_{2n}$), one sees that

$$\theta^2\|z_{2n}-x_{2n}\|^2 = \|x_{2n+1}-x_{2n}\|^2 . \tag{3.17}$$

Thus, we conclude from (3.16) that

$$\begin{aligned}
&\|w_{2n+1}-x^*\|^2 \\
&\leq \|x_{2n}-x^*\|^2-\theta(1+\alpha_{2n+1})\left(\frac{1}{2}\beta^*_{2n}+(1-\theta)-\alpha_{2n+1}\theta\right)\|x_{2n}-z_{2n}\|^2 .
\end{aligned} \tag{3.18}$$

Using (3.18) in (3.14), we have

$$\begin{aligned}\|x_{2n+2}-x^*\|^2 \leq \|x_{2n}-x^*\|^2 &- \theta\left(\frac{1}{2}\beta^*_{2n+1}+(1-\theta)\right)\|w_{2n+1}-z_{2n+1}\|^2 \\ &- \theta\left(1+\alpha_{2n+1}\right)\left(\frac{1}{2}\beta^*_{2n}+(1-\theta)-\alpha_{2n+1}\theta\right)\|x_{2n}-z_{2n}\|^2 .\end{aligned} \tag{3.19}$$

Since $\theta \in (0,1]$, $0 \leq \alpha_{2n+1} \leq \alpha < \frac{\beta^*+2(1-\theta)}{2\theta}$ and $\beta^*_{2n}, \beta^*_{2n+1} > 0$, $\forall n \geq N_0$, it follows from (3.19) that

$$\|x_{2n+2}-x^*\| \leq \|x_{2n}-x^*\|, \ \forall n \geq N_0.$$

This implies that the sequences $\{\|x_{2n}-x^*\|\}$ and $\{x_{2n}\}$ are bounded, and moreover $\lim_{n\to\infty}\|x_{2n}-x^*\|$ exists. In addition, we find from (3.19) that $\lim_{n\to\infty}\|x_{2n}-z_{2n}\| = 0$, and thus $\{z_{2n}\}$ is bounded. It follows that $\lim_{n\to\infty}\|x_{2n+1}-x_{2n}\| = 0$ by means of (3.17). In view of (3.13), one can show that (noting that $w_{2n} = x_{2n}$)

$$\|z_{2n}-x^*\|^2 \leq \|x_{2n}-x^*\|^2 - \beta^*_{2n}a_{2n},$$

which implies that

$$\begin{aligned}\beta^*_{2n}a_{2n} &\leq \|x_{2n}-x^*\|^2 - \|z_{2n}-x^*\|^2 \\ &= \left(\|x_{2n}-x^*\|+\|z_{2n}-x^*\|\right)\left(\|x_{2n}-x^*\|-\|z_{2n}-x^*\|\right) \\ &\leq \left(\|x_{2n}-x^*\|+\|z_{2n}-x^*\|\right)\|x_{2n}-z_{2n}\| \to 0, \quad \text{as } n\to\infty.\end{aligned}$$

Therefore, we infer that $\lim_{n\to\infty} a_{2n} = 0$, i.e.,

$$\lim_{n\to\infty}\|x_{2n}-y_{2n}\| = 0, \quad \lim_{n\to\infty}\|z_{2n}-y_{2n}\| = 0.$$

The proof is completed. □

Remark 3.4. Lemma 3.2 shows that unlike the inertial projection-type methods in [14, 12, 15, 39, 40], our proposed method yields Fejér monotonicity of the even iterative subsequence with respect to the solution.

We can easily obtain the following Lemma 3.3 by making a simple modification of Lemma 4.3 in [38]. To avoid repetitive expressions, we omit the proof here.

Lemma 3.3 ([38])**.** *Assume that $\{x_n\}$ is generated by Algorithm 3.1. Let $p \in \mathcal{H}$ denote the weak limit of the subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$. Then $p \in \mathrm{VI}(C, A)$.*

Remark 3.5. Notice that the proof of Lemma 3.3 is not necessary to impose the sequential weak continuity contained in Condition (C3) on the mapping A when it is monotone (see, e.g., [40, Remark 3.11]).

Now, we are in a position to proof the weak convergence of the proposed Algorithm 3.1.

Theorem 3.1. *Suppose that the sequence $\{x_n\}$ is generated by Algorithm 3.1 and Conditions (C1)–(C5) hold. Then $\{x_n\}$ converges weakly to a point in* $\mathrm{VI}(C, A)$.

Proof. It follows from Lemma 3.2 that $\{x_{2n}\}$ is bounded and thus $\{x_{2n}\}$ has weakly convergent subsequences. Suppose that $p \in \mathcal{H}$ represents the weak limit of such a subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$. Using Lemma 3.3, one obtains $p \in \mathrm{VI}(C, A)$. Meanwhile, we have that $\lim_{n\to\infty} \|x_{2n} - p\|$ exists by means of Lemma 3.2. Thus, from Lemma 2.2, one can show that the whole sequence $\{x_{2n}\}$ converges weakly to a point in $\mathrm{VI}(C, A)$. Next, we show that the weak limit is unique. Suppose that $\{x_{2n}\}$ converges weakly to $p \in \mathrm{VI}(C, A)$ and $\{x_{2n}\}$ converges weakly to $q \in \mathrm{VI}(C, A)$. Then

$$\begin{aligned}\|p - q\|^2 &= \langle p, p - q\rangle - \langle q, p - q\rangle \\ &= \lim_{n\to\infty} \langle x_{2n}, p - q\rangle - \lim_{n\to\infty} \langle x_{2n}, p - q\rangle \\ &= \lim_{n\to\infty} \langle x_{2n} - x_{2n}, p - q\rangle = 0.\end{aligned}$$

Hence, the weak limit p is unique. By definition, we obtain that $\lim\limits_{n\to\infty} \langle x_{2n} - p, z\rangle = 0$ for all $z \in \mathcal{H}$. Recalling that $\lim_{n\to\infty} \|x_{2n+1} - x_{2n}\| = 0$ in Lemma 3.2, we have for all $z \in \mathcal{H}$,

$$\begin{aligned}|\langle x_{2n+1} - p, z\rangle| &= |\langle x_{2n+1} - p + x_{2n} - x_{2n}, z\rangle| \\ &\le |\langle x_{2n} - p, z\rangle| + |\langle x_{2n+1} - x_{2n}, z\rangle| \\ &\le |\langle x_{2n} - p, z\rangle| + \|x_{2n+1} - x_{2n}\| \, \|z\| \to 0, \quad \text{as } n \to \infty,\end{aligned}$$

which implies that $\{x_{2n+1}\}$ also converges weakly to p. Thus, we conclude that the sequence $\{x_n\}$ converges weakly to a point $p \in \mathrm{VI}(C, A)$. The proof is completed. □

3.2. **The second algorithm.** In this subsection, we present another version of the suggested Algorithm 3.1. Specifically, our second alternated inertial subgradient extragradient scheme with relaxation effects and adaptive non-monotonic step size is shown in Algorithm 3.2 below.

Algorithm 3.2 The second modified subgradient extragradient method for (VIP).

Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:
Step 1. Compute w_n by (3.1).
Step 2. Compute $y_n = P_C\left(w_n - \beta\lambda_n A w_n\right)$, and update λ_{n+1} by (S2). If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = (1 - \theta)w_n + \theta z_n$, where $z_n = P_{H_n}(w_n - \lambda_n A y_n)$, and

$$H_n = \{x \in \mathcal{H} \mid \langle w_n - \beta\lambda_n A w_n - y_n, x - y_n\rangle \le 0\}\,. \tag{3.20}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.6. It should be emphasized that our Algorithm 3.2 in computing y_n and T_n is different from the proposed Algorithm 3.1 and the subgradient extragradient method introduced by Censor et al. [5, 6, 7]. The computational efficiency of Algorithm 3.2 is demonstrated in the numerical examples provided in Sect. 5. It is worth noting that the proposed Algorithm 3.1 and Algorithm 3.2 are equivalent when $\beta = 1$.

We assume that the proposed Algorithm 3.2 satisfies the following condition (C6) for the purpose of its weak convergence analysis.

(C6) Let $\theta \in (0,1]$, $\beta \in (1/(2-\mu), 1/\mu)$, $0 \leq \alpha_n \leq \alpha < \frac{\beta^\dagger + 2(1-\theta)}{2\theta}$, where $\beta^\dagger = 2 - \frac{1}{\beta} - \mu$ when $\beta \in (0,1]$ and $\beta^\dagger = \frac{1}{\beta} - \mu$ when $\beta > 1$.

Remark 3.7. Note that the inertial parameter α_n in Condition (C6) is allowed to be greater than or equal to 1 when the relaxation parameter $\theta \in (0,1)$; e.g., by choosing $\theta = 0.5$, then $\alpha_n \leq \alpha < \beta^\dagger + 1$ ($\beta^\dagger > 0$ for all $\beta \in (1/(2-\mu), 1/\mu)$).

Similar to the proof of Lemma 3.2, we can obtain the following Lemma 3.4, which is essential for the convergence analysis of Algorithm3.2.

Lemma 3.4. *Assume that Conditions (C2) and (C6) hold, and the sequence* $\{x_n\}$ *is created by Algorithm 3.2. Then the sequence* $\{x_{2n}\}$ *is Fejér monotone with respect to* $\mathrm{VI}(C,A)$ *and* $\lim_{n\to\infty} \|x_{2n} - x^*\|$ *exists, where* $x^* \in \mathrm{VI}(C,A)$. *Moreover,* $\lim_{n\to\infty} \|x_{2n} - y_{2n}\| = 0$ *and* $\lim_{n\to\infty} \|x_{2n+1} - x_{2n}\| = 0$.

Proof. Using (3.3) and (3.4), we have

$$\begin{aligned} \|z_{2n+1} - x^*\|^2 \leq \|w_{2n+1} - x^*\|^2 - \|w_{2n+1} - z_{2n+1}\|^2 \\ - 2\langle \lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle . \end{aligned} \tag{3.21}$$

Now we estimate $2\langle \lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle$. Note that

$$\begin{aligned} - \|w_{2n+1} - z_{2n+1}\|^2 = - \|w_{2n+1} - y_{2n+1}\|^2 - \|y_{2n+1} - z_{2n+1}\|^2 \\ + 2\langle w_{2n+1} - y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle . \end{aligned} \tag{3.22}$$

In addition,

$$\begin{aligned} & \langle w_{2n+1} - y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ = & \langle w_{2n+1} - y_{2n+1} - \beta\lambda_{2n+1} A w_{2n+1} + \beta\lambda_{2n+1} A w_{2n+1} \\ & -\beta\lambda_{2n+1} A y_{2n+1} + \beta\lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ = & \langle w_{2n+1} - \beta\lambda_{2n+1} A w_{2n+1} - y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ & + \beta\lambda_{2n+1} \langle A w_{2n+1} - A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ & + \langle \beta\lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle . \end{aligned} \tag{3.23}$$

Since $z_{2n+1} \in H_{2n+1}$, one has

$$\langle w_{2n+1} - \beta\lambda_{2n+1} A w_{2n+1} - y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \leq 0. \tag{3.24}$$

Putting (3.8), (3.23) and (3.24) into (3.22), we arrive at

$$\begin{aligned} & - \|w_{2n+1} - z_{2n+1}\|^2 \\ \leq & 2\beta \langle \lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ & - \left(1 - \frac{\beta\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}\right) (\underbrace{\|w_{2n+1} - y_{2n+1}\|^2 + \|z_{2n+1} - y_{2n+1}\|^2}_{a_{2n+1}}), \end{aligned}$$

which implies that

$$\begin{aligned} & - 2\langle \lambda_{2n+1} A y_{2n+1}, z_{2n+1} - y_{2n+1} \rangle \\ \leq & - \left(\frac{1}{\beta} - \frac{\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}\right) a_{2n+1} + \frac{1}{\beta} \|w_{2n+1} - z_{2n+1}\|^2 . \end{aligned} \tag{3.25}$$

Combining (3.21) and (3.25), we obtain

$$\begin{aligned}\|z_{2n+1}-x^*\|^2 \leq \|w_{2n+1}-x^*\|^2 - \left(\frac{1}{\beta}-\frac{\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}\right)a_{2n+1} \\ -\left(1-\frac{1}{\beta}\right)\|w_{2n+1}-z_{2n+1}\|^2.\end{aligned} \tag{3.26}$$

Note that $\|w_{2n+1}-z_{2n+1}\|^2 \leq 2a_{2n+1}$, which yields

$$-\left(1-\frac{1}{\beta}\right)\|w_{2n+1}-z_{2n+1}\|^2 \leq -2\left(1-\frac{1}{\beta}\right)a_{2n+1},\ \forall \beta \in (0,1].$$

This together with (3.26) implies that

$$\|z_{2n+1}-x^*\|^2 \leq \|w_{2n+1}-x^*\|^2 - \left(2-\frac{1}{\beta}-\frac{\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}\right)a_{2n+1},\ \forall \beta \in (0,1].$$

On the other hand, if $\beta > 1$, then

$$\|z_{2n+1}-x^*\|^2 \leq \|w_{2n+1}-x^*\|^2 - \left(\frac{1}{\beta}-\frac{\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}\right)a_{2n+1},\ \forall \beta > 1.$$

Thus, we conclude that

$$\|z_{2n+1}-x^*\|^2 \leq \|w_{2n+1}-x^*\|^2 - \beta^\dagger_{2n+1}a_{2n+1},$$

where $\beta^\dagger_{2n+1} = 2-\frac{1}{\beta}-\frac{\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}$ when $\beta \in (0,1]$ and $\beta^\dagger_{2n+1} = \frac{1}{\beta}-\frac{\mu q_{2n+1}\lambda_{2n+1}}{\lambda_{2n+2}}$ when $\beta > 1$. By Condition (C4) and Lemma 3.1, we have

$$\beta^\dagger := \lim_{n\to\infty}\beta^\dagger_{2n+1} = \begin{cases} 2-\frac{1}{\beta}-\mu, & \beta \in (1/(2-\mu),1]; \\ \frac{1}{\beta}-\mu, & \beta \in (1,1/\mu). \end{cases}$$

Hence, $\lim_{n\to\infty}\beta^\dagger_{2n+1} > 0$ for all $\beta \in (1/(2-\mu),1/\mu)$. There exists a positive constant N_1 such that $\beta^\dagger_{2n+1} > 0$ holds for all $n \geq N_1$.
With the help of the proof of Lemma 3.2, we can easily obtain

$$\begin{aligned}\|x_{2n+2}-x^*\|^2 \leq \|x_{2n}-x^*\|^2 - \theta\left(\frac{1}{2}\beta^\dagger_{2n+1}+(1-\theta)\right)\|w_{2n+1}-z_{2n+1}\|^2 \\ -\theta\left(1+\alpha_{2n+1}\right)\left(\frac{1}{2}\beta^\dagger_{2n}+(1-\theta)-\alpha_{2n+1}\theta\right)\|x_{2n}-z_{2n}\|^2.\end{aligned} \tag{3.27}$$

Since $\theta \in (0,1]$, $0 \leq \alpha_{2n+1} \leq \alpha < \frac{\beta^\dagger+2(1-\theta)}{2\theta}$ and $\beta^\dagger_{2n}, \beta^\dagger_{2n+1} > 0$, $\forall n \geq N_1$, we see from (3.27) that

$$\|x_{2n+2}-x^*\| \leq \|x_{2n}-x^*\|,\ \forall n \geq N_1.$$

This implies that $\{x_{2n}\}$ is bounded and that $\lim_{n\to\infty}\|x_{2n}-x^*\|$ exists. Moreover, one can show that $\lim_{n\to\infty}\|x_{2n}-y_{2n}\| = 0$ and $\lim_{n\to\infty}\|x_{2n+1}-x_{2n}\| = 0$ by means of Lemma 3.2. This completes the proof. □

Similarly, we can obtain the following Lemma 3.5 by performing a simple modification of Lemma 4.3 in [38]. We omit the proof to avoid redundancy.

Lemma 3.5 ([38]). *Assume that the sequence $\{x_n\}$ is created by Algorithm 3.2. Let $p \in \mathcal{H}$ denote the weak limit of the subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$. Then $p \in \mathrm{VI}(C, A)$.*

Theorem 3.2. *Suppose that the sequence $\{x_n\}$ is created by Algorithm 3.2 and Conditions (C1)–(C4) and (C6) hold. Then $\{x_n\}$ converges weakly to a point in $\mathrm{VI}(C, A)$.*

Proof. From Lemma 3.4 and Theorem 3.1, we can easily obtain the conclusion required. □

3.3. **The third algorithm.** In this subsection, inspired by the work in [29, 40, 38], we present the last adaptive alternated inertial iterative scheme with relaxation effects proposed in this paper, which is shown in Algorithm 3.3 below.

Algorithm 3.3 The modified projection and contraction method for (VIP).

Iterative Steps: Let $x_0, x_1 \in \mathcal{H}$. Calculate x_{n+1} as follows:
Step 1. Compute w_n by (3.1).
Step 2. Compute $y_n = P_C(w_n - \beta\lambda_n A w_n)$, and update λ_{n+1} by (S2). If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1} = (1-\theta)w_n + \theta z_n$, where $z_n = w_n - \gamma\eta_n d_n$, and

$$\eta_n = \begin{cases} \dfrac{\langle w_n - y_n, d_n\rangle}{\|d_n\|^2}, & d_n \neq 0; \\ 0, & d_n = 0, \end{cases} \quad \text{and} \quad d_n = w_n - y_n - \beta\lambda_n (A w_n - A y_n). \tag{3.28}$$

Set $n := n+1$ and go to **Step 1**.

Remark 3.8. Our Algorithm 3.3 improves the results in the literature [12, 40, 38] based on the following observations: (1) our Algorithm 3.3 allows the inertial parameter $\alpha_n \geq 1$, which is not permitted in the Algorithm 3.1 suggested by Dong et al. [12] and the Algorithm 3.1 proposed by Shehu et al. [40]; (2) the even subsequence generated by our Algorithm 3.3 is Fejér monotone with respect to the solution, while this property is not enjoyed in the algorithms presented in [12, 40]; (3) our Algorithm 3.3 is different from the algorithms presented in [12, 40, 38] in the computation of y_n, η_n, d_n and z_n due to the fact that the insertion of a new parameter β; (4) the Algorithm 3.1 introduced in [12] utilizes a fixed-step and the algorithms presented in [40, 38] use a non-increasing step size criterion (S1), while our Algorithm 3.3 employs a non-monotonic step size criterion (S2), which is preferable in practical applications; (5) when $\beta = \theta = 1$, $q_n = \xi_n = 1$ and $\zeta_n = 0$ in our Algorithm 3.3, it degenerates to the Algorithm 2 introduced in [38]; and (6) the operator A of our proposed Algorithm 3.3 is pseudo-monotone, so it is more useful than the Algorithm 3.1 of Dong et al. [12], where operator A is assumed to be monotone. Therefore, our Algorithm 3.3 is more useful and efficient than the algorithms in [12, 40, 38].

The proposed Algorithm 3.3 is assumed to meet the following condition (C7) in order to perform its convergence analysis.

(C7) Let $\theta \in (0, 1]$, $\gamma \in (0, 2)$, $0 \leq \alpha_n \leq \alpha < \frac{2}{\gamma\theta} - 1$, and $\beta \in (0, 1/\mu)$.

Remark 3.9. It is important to note that the inertial parameter in Condition (C7) allows $\alpha_n \geq 1$ when $\gamma\theta < 1$.

The establishment of the following two lemmas is crucial for the convergence analysis of the proposed Algorithm 3.3.

Lemma 3.6. *Assume that Conditions (C2) and (C7) hold, and the sequence $\{x_n\}$ is formed by Algorithm 3.3. Then the sequence $\{x_{2n}\}$ is Fejér monotone with respect to* $\mathrm{VI}(C, A)$ *and* $\lim_{n\to\infty} \|x_{2n} - x^*\|$ *exists, where* $x^* \in \mathrm{VI}(C, A)$. *Moreover,* $\lim_{n\to\infty} \|x_{2n} - y_{2n}\| = 0$ *and* $\lim_{n\to\infty} \|x_{2n+1} - x_{2n}\| = 0$.

Proof. From the definition of z_{2n+1}, one obtains

$$\begin{aligned}\|z_{2n+1} - x^*\|^2 &= \|w_{2n+1} - \gamma\eta_{2n+1} d_{2n+1} - x^*\|^2 \\ &= \|w_{2n+1} - x^*\|^2 + \gamma^2\eta_{2n+1}^2 \|d_{2n+1}\|^2 \\ &\quad - 2\gamma\eta_{2n+1} \langle w_{2n+1} - x^*, d_{2n+1}\rangle .\end{aligned} \tag{3.29}$$

Note that

$$\langle w_{2n+1} - x^*, d_{2n+1}\rangle = \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle + \langle y_{2n+1} - x^*, d_{2n+1}\rangle . \tag{3.30}$$

By $y_{2n+1} = P_C(w_{2n+1} - \beta\lambda_{2n+1} A w_{2n+1})$ and (2.2), we have

$$\langle w_{2n+1} - y_{2n+1} - \beta\lambda_{2n+1} A w_{2n+1}, y_{2n+1} - x^*\rangle \geq 0. \tag{3.31}$$

Using $x^* \in \mathrm{VI}(C, A)$ and $y_{2n+1} \in C$, we have $\langle Ax^*, y_{2n+1} - x^*\rangle \geq 0$, which together with the pseudo-monotonicity of mapping A yields that $\langle Ay_{2n+1}, y_{2n+1} - x^*\rangle \geq 0$. Thus

$$\langle \beta\lambda_{2n+1} A y_{2n+1}, y_{2n+1} - x^*\rangle \geq 0. \tag{3.32}$$

Adding (3.31) and (3.32), we obtain

$$\langle w_{2n+1} - y_{2n+1} - \beta\lambda_{2n+1}(Aw_{2n+1} - Ay_{2n+1}), y_{2n+1} - x^*\rangle \geq 0,$$

i.e.,

$$\langle y_{2n+1} - x^*, d_{2n+1}\rangle \geq 0. \tag{3.33}$$

Combining (3.30) and (3.33), we deduce that

$$\langle w_{2n+1} - x^*, d_{2n+1}\rangle \geq \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle . \tag{3.34}$$

Putting (3.34) into (3.29) (noting that $\eta_{2n+1} \|d_{2n+1}\|^2 = \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle$), we have

$$\begin{aligned}&\|z_{2n+1} - x^*\|^2 \\ &\leq \|w_{2n+1} - x^*\|^2 - 2\gamma\eta_{2n+1} \langle w_{2n+1} - y_{2n+1}, d_{2n+1}\rangle + \gamma^2\eta_{2n+1}^2 \|d_{2n+1}\|^2 \\ &= \|w_{2n+1} - x^*\|^2 - 2\gamma\eta_{2n+1}^2 \|d_{2n+1}\|^2 + \gamma^2\eta_{2n+1}^2 \|d_{2n+1}\|^2 \\ &= \|w_{2n+1} - x^*\|^2 - \frac{2-\gamma}{\gamma}\|\gamma\eta_{2n+1} d_{2n+1}\|^2.\end{aligned}$$

This together with the definition of z_{2n+1} yields

$$\|z_{2n+1} - x^*\|^2 \leq \|w_{2n+1} - x^*\|^2 - \frac{2-\gamma}{\gamma} \|z_{2n+1} - w_{2n+1}\|^2 . \tag{3.35}$$

By the definition of x_{2n+2}, one concludes

$$\left\|z_{2n+1}-w_{2n+1}\right\|^{2}=\frac{1}{\theta^{2}}\left\|x_{2n+2}-w_{2n+1}\right\|^{2}. \tag{3.36}$$

Using (2.1), (3.35), and (3.36), one has

$$\begin{aligned}\left\|x_{2n+2}-x^{*}\right\|^{2} &= \left\|(1-\theta)\left(w_{2n+1}-x^{*}\right)+\theta\left(z_{2n+1}-x^{*}\right)\right\|^{2} \\ &= (1-\theta)\left\|w_{2n+1}-x^{*}\right\|^{2}+\theta\left\|z_{2n+1}-x^{*}\right\|^{2} \\ &\quad -\theta(1-\theta)\left\|w_{2n+1}-z_{2n+1}\right\|^{2} \\ &\leq (1-\theta)\left\|w_{2n+1}-x^{*}\right\|^{2}+\theta\left\|w_{2n+1}-x^{*}\right\|^{2} \\ &\quad -\theta\frac{2-\gamma}{\gamma}\left\|z_{2n+1}-w_{2n+1}\right\|^{2}-\theta(1-\theta)\left\|w_{2n+1}-z_{2n+1}\right\|^{2} \\ &= \left\|w_{2n+1}-x^{*}\right\|^{2}-\left(\frac{2}{\gamma\theta}-1\right)\left\|x_{2n+2}-w_{2n+1}\right\|^{2}.\end{aligned} \tag{3.37}$$

From (3.37) (noting that $w_{2n}=x_{2n}$), we infer that

$$\begin{aligned}\left\|x_{2n+1}-x^{*}\right\|^{2} &\leq \left\|w_{2n}-x^{*}\right\|^{2}-\left(\frac{2}{\gamma\theta}-1\right)\left\|x_{2n+1}-w_{2n}\right\|^{2} \\ &= \left\|x_{2n}-x^{*}\right\|^{2}-\left(\frac{2}{\gamma\theta}-1\right)\left\|x_{2n+1}-x_{2n}\right\|^{2}.\end{aligned} \tag{3.38}$$

Combining (2.1) and (3.38), we obtain

$$\begin{aligned} &\left\|w_{2n+1}-x^{*}\right\|^{2} \\ &= \left\|\left(1+\alpha_{2n+1}\right)\left(x_{2n+1}-x^{*}\right)-\alpha_{2n+1}\left(x_{2n}-x^{*}\right)\right\|^{2} \\ &= \left(1+\alpha_{2n+1}\right)\left\|x_{2n+1}-x^{*}\right\|^{2}-\alpha_{2n+1}\left\|x_{2n}-x^{*}\right\|^{2} \\ &\quad +\alpha_{2n+1}\left(1+\alpha_{2n+1}\right)\left\|x_{2n+1}-x_{2n}\right\|^{2} \\ &\leq \left(1+\alpha_{2n+1}\right)\left[\left\|x_{2n}-x^{*}\right\|^{2}-\left(\frac{2}{\gamma\theta}-1\right)\left\|x_{2n+1}-x_{2n}\right\|^{2}\right] \\ &\quad -\alpha_{2n+1}\left\|x_{2n}-x^{*}\right\|^{2}+\alpha_{2n+1}\left(1+\alpha_{2n+1}\right)\left\|x_{2n+1}-x_{2n}\right\|^{2} \\ &= \left\|x_{2n}-x^{*}\right\|^{2}-\left(1+\alpha_{2n+1}\right)\left(\frac{2}{\gamma\theta}-1-\alpha_{2n+1}\right)\left\|x_{2n+1}-x_{2n}\right\|^{2}.\end{aligned} \tag{3.39}$$

Using (3.39) in (3.37), we have

$$\begin{aligned}\left\|x_{2n+2}-x^{*}\right\|^{2} \leq &\left\|x_{2n}-x^{*}\right\|^{2}-\left(\frac{2}{\gamma\theta}-1\right)\left\|x_{2n+2}-w_{2n+1}\right\|^{2} \\ &-\left(1+\alpha_{2n+1}\right)\left(\frac{2}{\gamma\theta}-1-\alpha_{2n+1}\right)\left\|x_{2n+1}-x_{2n}\right\|^{2}.\end{aligned} \tag{3.40}$$

Since $\gamma\in(0,2)$, $\theta\in(0,1]$ and $0\leq\alpha_{2n+1}\leq\alpha<\frac{2}{\gamma\theta}-1$, it follows from (3.40) that

$$\left\|x_{2n+2}-x^{*}\right\|\leq\left\|x_{2n}-x^{*}\right\|,\ \forall n\geq 1.$$

This implies that the sequence $\{\|x_{2n} - x^*\|\}$ and $\{x_{2n}\}$ are bounded. Furthermore, one obtains $\lim_{n\to\infty} \|x_{2n} - x^*\|$ exists. Rearranging (3.40) and using the fact that $\{\|x_{2n} - x^*\|\}$ is bounded, we have

$$\lim_{n\to\infty} \|x_{2n+1} - x_{2n}\| = 0. \tag{3.41}$$

From the definition of d_{2n} and (S2), we obtain

$$\begin{aligned}
\|d_{2n}\| &= \|w_{2n} - y_{2n} - \beta\lambda_{2n}(Aw_{2n} - Ay_{2n})\| \\
&\le \|w_{2n} - y_{2n}\| + \beta\lambda_{2n}\|Aw_{2n} - Ay_{2n}\| \\
&\le \left(1 + \frac{\beta q_{2n}\mu\lambda_{2n}}{\lambda_{2n+1}}\right)\|w_{2n} - y_{2n}\|,
\end{aligned}$$

which implies that

$$\frac{1}{\|d_{2n}\|} \ge \frac{1}{\left(1 + \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}\right)\|w_{2n} - y_{2n}\|}. \tag{3.42}$$

From the definition of d_{2n} and (S2), one has

$$\begin{aligned}
\langle w_{2n} - y_{2n}, d_{2n}\rangle &= \langle w_{2n} - y_{2n}, w_{2n} - y_{2n} - \beta\lambda_{2n}(Aw_{2n} - Ay_{2n})\rangle \\
&= \|w_{2n} - y_{2n}\|^2 - \langle w_{2n} - y_{2n}, \beta\lambda_{2n}(Aw_{2n} - Ay_{2n})\rangle \\
&\ge \|w_{2n} - y_{2n}\|^2 - \beta\lambda_{2n}\|Aw_{2n} - Ay_{2n}\|\|w_{2n} - y_{2n}\| \\
&\ge \|w_{2n} - y_{2n}\|^2 - \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}\|w_{2n} - y_{2n}\|^2 \\
&= \left(1 - \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}\right)\|w_{2n} - y_{2n}\|^2.
\end{aligned} \tag{3.43}$$

Combining the definition of x_{2n+1} and η_{2n}, (3.42) and (3.43), we have

$$\begin{aligned}
\|x_{2n+1} - w_{2n}\| &= \|\theta(z_{2n} - w_{2n})\| = \theta\gamma\eta_{2n}\|d_{2n}\| \\
&= \theta\gamma\frac{\langle w_{2n} - y_{2n}, d_{2n}\rangle}{\|d_{2n}\|} \ge \theta\gamma\left(\frac{1 - \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}{1 + \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}\right)\|w_{2n} - y_{2n}\|.
\end{aligned} \tag{3.44}$$

Using Condition (C4), Lemma 3.1 and $\beta \in (0, 1/\mu)$, we obtain

$$\lim_{n\to\infty} \frac{1 - \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}{1 + \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}} = \lim_{n\to\infty} \frac{1 - \beta\mu}{1 + \beta\mu} > 0.$$

By (3.41) and (3.44) (noting that $w_{2n} = x_{2n}$), we deduce that

$$\lim_{n\to\infty} \|x_{2n} - y_{2n}\| = 0.$$

The proof is completed. □

We can also obtain the following Lemma 3.7 by a simple modification of Lemma 4.3 in [38], so that the proof is omitted.

Lemma 3.7 ([38]). *Assume that the sequence $\{x_n\}$ is formed by Algorithm 3.3. Let $p \in \mathcal{H}$ denote the weak limit of the subsequence $\{x_{2n_k}\}$ of $\{x_{2n}\}$. Then $p \in \mathrm{VI}(C, A)$.*

Theorem 3.3. *Suppose that the sequence $\{x_n\}$ is formed by Algorithm 3.3 and Conditions (C1)–(C4) and (C7) hold. Then $\{x_n\}$ converges weakly to a point in* $\mathrm{VI}(C, A)$.

Proof. We can easily prove the theorem by combining Theorem 3.1 and Lemma 3.6. □

4. Linear convergence rate

In this section, we perform linear convergence rate analysis for the proposed Algorithms 3.1–3.3 under the condition that the operator A is δ-strongly pseudo-monotone. Therefore, we need to replace the previous Condition (C3) with the following Condition (C8).

(C8) The mapping $A : \mathcal{H} \to \mathcal{H}$ is δ-strongly pseudo-monotone, L-Lipschitz continuous and sequentially weakly continuous on bounded subsets of $\mathcal{H}$.

Now, we are in a position to prove the following theorems with the help of the techniques in [29, 40, 38].

Theorem 4.1. *Let the sequence $\{x_n\}$ be generated by Algorithm 3.1. If Conditions (C1), (C2), (C4), (C5) and (C8) hold, then $\{x_n\}$ converges at least R-linearly to the unique solution x^* of* (VIP).

Proof. From the definition of y_{2n}, $x^* \in C$ and (2.2), we obtain

$$\langle w_{2n} - \lambda_{2n} A w_{2n} - y_{2n}, x^* - y_{2n} \rangle \leq 0. \tag{4.1}$$

By $x^* \in \mathrm{VI}(C, A)$ and $y_{2n} \in C$, we have $\langle Ax^*, y_{2n} - x^* \rangle \geq 0$. This together with the δ-strongly pseudo-monotonicity of mapping A implies that

$$\langle A y_{2n}, y_{2n} - x^* \rangle \geq \delta \left\| x^* - y_{2n} \right\|^2 . \tag{4.2}$$

Combining (S2), (4.1) and (4.2), we have

$$\begin{aligned} \langle w_{2n} - y_{2n}, x^* - y_{2n} \rangle &\leq \lambda_{2n} \langle A w_{2n}, x^* - y_{2n} \rangle \\ &= \lambda_{2n} \langle A w_{2n} - A y_{2n}, x^* - y_{2n} \rangle + \lambda_{2n} \langle A y_{2n}, x^* - y_{2n} \rangle \\ &\leq \lambda_{2n} \langle A w_{2n} - A y_{2n}, x^* - y_{2n} \rangle - \delta \lambda_{2n} \left\| x^* - y_{2n} \right\|^2 \\ &\leq \frac{\mu q_{2n} \lambda_{2n}}{\lambda_{2n+1}} \left\| w_{2n} - y_{2n} \right\| \left\| x^* - y_{2n} \right\| - \delta \lambda_{2n} \left\| x^* - y_{2n} \right\|^2 , \end{aligned}$$

which implies that

$$\begin{aligned} \delta \lambda_{2n} \left\| x^* - y_{2n} \right\|^2 &\leq \frac{\mu q_{2n} \lambda_{2n}}{\lambda_{2n+1}} \left\| w_{2n} - y_{2n} \right\| \left\| x^* - y_{2n} \right\| + \langle w_{2n} - y_{2n}, y_{2n} - x^* \rangle \\ &\leq \frac{\mu q_{2n} \lambda_{2n}}{\lambda_{2n+1}} \left\| w_{2n} - y_{2n} \right\| \left\| x^* - y_{2n} \right\| + \left\| w_{2n} - y_{2n} \right\| \left\| x^* - y_{2n} \right\| . \end{aligned}$$

Thus

$$\left\| x^* - y_{2n} \right\| \leq \frac{1 + \frac{\mu q_{2n} \lambda_{2n}}{\lambda_{2n+1}}}{\delta \lambda_{2n}} \left\| w_{2n} - y_{2n} \right\| .$$

Let $\Delta_{2n} = \frac{1 + \frac{\mu q_{2n} \lambda_{2n}}{\lambda_{2n+1}}}{\delta \lambda_{2n}}$. Therefore

$$\left\| w_{2n} - x^* \right\| \leq \left\| w_{2n} - y_{2n} \right\| + \left\| x^* - y_{2n} \right\| \leq (1 + \Delta_{2n}) \left\| w_{2n} - y_{2n} \right\| . \tag{4.3}$$

By (4.3) (noting that $w_{2n} = x_{2n}$), one sees that

$$\|x_{2n} - y_{2n}\| \geq (1 + \Delta_{2n})^{-1} \|x_{2n} - x^*\| . \tag{4.4}$$

From the definition of a_{2n} in (3.9) and the inequality $\|a\|^2 + \|b\|^2 \leq \|a + b\|^2$ (noting that $w_{2n} = x_{2n}$), we obtain

$$a_{2n} = \|w_{2n} - y_{2n}\|^2 + \|z_{2n} - y_{2n}\|^2 \leq \|w_{2n} - z_{2n}\|^2 = \|x_{2n} - z_{2n}\|^2 . \tag{4.5}$$

Combining (3.19), (4.4) and (4.5), we have

$$\begin{aligned} &\|x_{2n+2} - x^*\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \underbrace{\theta\left(\frac{1}{2}\beta^*_{2n} + (1-\theta) - \alpha_{2n+1}\theta\right)}_{\Gamma_{2n}} \|x_{2n} - z_{2n}\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \Gamma_{2n} a_{2n} \\ &\leq \|x_{2n} - x^*\|^2 - \Gamma_{2n} \|x_{2n} - y_{2n}\|^2 \\ &\leq \left[1 - \Gamma_{2n} (1 + \Delta_{2n})^{-2}\right] \|x_{2n} - x^*\|^2 . \end{aligned} \tag{4.6}$$

Note that $\Gamma_{2n} (1 + \Delta_{2n})^{-2} > 0$, $\forall n > N_0$ and thus $1 - \Gamma_{2n} (1 + \Delta_{2n})^{-2} := \sigma < 1$, $\forall n > N_0$. It follows from (4.6) that

$$\begin{aligned} \|x_{2n+2} - x^*\|^2 &\leq \sigma \|x_{2n} - x^*\|^2 \leq \sigma^2 \|x_{2n-2} - x^*\|^2 \\ &\leq \cdots \leq \sigma^{(n+1)-N} \|x_{2N} - x^*\|^2 , \ \forall n > N_0, \end{aligned}$$

which implies that

$$\|x_{2n} - x^*\|^2 \leq \frac{\|x_{2N} - x^*\|^2}{\sigma^N} \sigma^n, \ \forall n \geq N_0. \tag{4.7}$$

From (3.15) and (4.6), we have

$$\begin{aligned} \|x_{2n+1} - x^*\|^2 &\leq \|x_{2n} - x^*\|^2 - \theta\left(\frac{1}{2}\beta^*_{2n} + (1-\theta)\right) \|x_{2n} - z_{2n}\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \theta\left(\frac{1}{2}\beta^*_{2n} + (1-\theta) - \alpha_{2n+1}\theta\right) \|x_{2n} - z_{2n}\|^2 \\ &\leq \left[1 - \Gamma_{2n} (1 + \Delta_{2n})^{-2}\right] \|x_{2n} - x^*\|^2 , \end{aligned}$$

which combining with (4.7) yields

$$\begin{aligned} \|x_{2n+1} - x^*\|^2 &\leq \sigma \|x_{2n} - x^*\|^2 \\ &\leq \|x_{2n} - x^*\|^2 \leq \frac{\|x_{2N} - x^*\|^2}{\sigma^N} \sigma^n, \ \forall n \geq N_0. \end{aligned} \tag{4.8}$$

Thus, we deduce that $\{x_n\}$ converges R-Linearly to x^* by means of (4.7) and (4.8). The proof is completed. □

Theorem 4.2. *Let the sequence $\{x_n\}$ be created by Algorithm 3.2. If Conditions (C1), (C2), (C4), (C6) and (C8) hold, then $\{x_n\}$ converges at least R-linearly to the unique solution x^* of* (VIP)*.*

Proof. Combining Lemma 3.3 and Theorem 4.1, we can easily obtain the conclusion required. Therefore, we omit the details of the proof. □

Theorem 4.3. *Let the sequence $\{x_n\}$ be formed by Algorithm 3.3. If Conditions (C1), (C2), (C4), (C7) and (C8) hold, then $\{x_n\}$ converges at least R-linearly to the unique solution x^* of* (VIP)*.*

Proof. By (3.40) (noting that $\frac{2}{\gamma\theta} - 1 > 0$ and $\frac{2}{\gamma\theta} - 1 - \alpha_{2n+1} > 0$), one has

$$\|x_{2n+2} - x^*\|^2 \leq \|x_{2n} - x^*\|^2 - \left(\frac{2}{\gamma\theta} - 1 - \alpha_{2n+1}\right)\|x_{2n+1} - x_{2n}\|^2 . \tag{4.9}$$

Combining (3.44) and (4.9) (noting that $w_{2n} = x_{2n}$), we obtain

$$\begin{aligned} &\|x_{2n+2} - x^*\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \left(\frac{2}{\gamma\theta} - 1 - \alpha_{2n+1}\right)\|x_{2n+1} - x_{2n}\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \underbrace{\theta^2\gamma^2\left(\frac{1 - \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}{1 + \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}\right)^2\left(\frac{2}{\gamma\theta} - 1 - \alpha_{2n+1}\right)}_{\Upsilon_n}\|x_{2n} - y_{2n}\|^2 . \end{aligned} \tag{4.10}$$

From (4.4) and (4.10), we have

$$\begin{aligned} \|x_{2n+2} - x^*\|^2 &\leq \|x_{2n} - x^*\|^2 - \Upsilon_{2n}\|x_{2n} - y_{2n}\|^2 \\ &\leq \left[1 - \Upsilon_{2n}(1 + \Delta_{2n})^{-2}\right]\|x_{2n} - x^*\|^2 . \end{aligned} \tag{4.11}$$

Note that $\Upsilon_{2n}(1 + \Delta_{2n})^{-2} > 0$, $\forall n > 1$ and thus $1 - \Upsilon_{2n}(1 + \Delta_{2n})^{-2} := \rho < 1$, $\forall n > 1$. It follows from (4.11) that

$$\begin{aligned} \|x_{2n+2} - x^*\|^2 &\leq \rho\|x_{2n} - x^*\|^2 \leq \rho^2\|x_{2n-2} - x^*\|^2 \\ &\leq \cdots \leq \rho^n\|x_2 - x^*\|^2 , \quad \forall n \geq 1, \end{aligned}$$

which implies that

$$\|x_{2n} - x^*\|^2 \leq \frac{\|x_2 - x^*\|^2}{\rho}\rho^n, \quad \forall n \geq 1. \tag{4.12}$$

Using (3.38), (3.44) and (4.4), we deduce that

$$\begin{aligned} \|x_{2n+1} - x^*\|^2 &\leq \|x_{2n} - x^*\|^2 - \theta^2\gamma^2\left(\frac{1 - \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}{1 + \frac{\beta\mu q_{2n}\lambda_{2n}}{\lambda_{2n+1}}}\right)\left(\frac{2}{\gamma\theta} - 1\right)\|x_{2n} - y_{2n}\|^2 \\ &\leq \|x_{2n} - x^*\|^2 - \Upsilon_{2n}\|x_{2n} - y_{2n}\|^2 \\ &\leq \left[1 - \Upsilon_{2n}(1 + \Delta_{2n})^{-2}\right]\|x_{2n} - x^*\|^2 , \end{aligned}$$

which together with (4.12) yields that

$$\begin{aligned}\|x_{2n+1}-x^*\|^2 &\le \rho\,\|x_{2n}-x^*\|^2\\ &\le \|x_{2n}-x^*\|^2 \le \frac{\|x_2-x^*\|^2}{\rho}\rho^n,\ \forall n\ge 1.\end{aligned} \tag{4.13}$$

Thus, from (4.12) and (4.13), we conclude that $\{x_n\}$ converges R-Linearly to x^*. This completes the proof. □

Remark 4.1. We summarize our contributions in this paper as follows.

(1) We propose three adaptive relaxed projection methods with alternating inertial extrapolation steps to solve variational inequalities in infinite-dimensional Hilbert spaces. The proposed algorithms have a significant computational advantage over the extragradient-type methods (see, e.g., [14, 23]) when computing the projection onto the feasible set is difficult, which is due to the fact that our algorithms require computing the projection onto the feasible set only once in each iteration, instead of twice as required toward the extragradient-type methods in [14, 23].

(2) Our alternated inertial projection algorithms differ from the inertial projection-type methods in the literature [14, 12, 15, 39, 40]. Our algorithms can recover the Fejér monotonicity of the even subsequence with respect to the solution, while the inertia-type methods proposed in [14, 12, 15, 39, 40] do not enjoy this property. Furthermore, it is worth noting that our relaxed methods allow inertial parameters $\alpha_n \ge 1$. This is not available in many known (alternated) inertial projection-type methods; see, for example, the algorithms in [14, 12, 15, 29, 39] that require $\alpha_n < 1$.

(3) Our three iterative schemes can be applied to pseudo-monotone variational inequality problems, which extends many results in the literature (see, e.g., [13, 14, 12, 16, 15, 37, 39]) for solving monotone variational inequalities. Thus, our methods have a wide range of applications. In addition, the R-linear convergence rates of the proposed algorithms are proved under the assumption that the operator A is strongly pseudo-monotone.

(4) The suggested methods contain some known results in the literature [5, 6, 7, 17, 38]. For example, when $\alpha = 0$, $\beta = 1$ and $\theta = 1$ in the proposed Algorithm 3.1 (or Algorithm 3.2) and Algorithm 3.3, they degenerate to the subgradient extragradient method introduced by Censor et al. [5, 6, 7] and the projection and contraction method proposed by He [17], respectively. Moreover, our Algorithm 3.3 with $\beta = 1$ is a relaxed version of the Algorithm 2 proposed by Shehu and Iyiola [38].

(5) Our methods use a non-monotonic step size criterion, which makes them more efficient than the methods presented in [15, 29, 39, 40, 38, 41] that use a non-increasing step size, the methods introduced in [13, 16, 37] that apply an Armijo-type step size, and the fixed-step methods offered in [14, 12, 34]. We provide some numerical experiments and applications to show that the proposed methods have a competitive advantage over some (alternated) inertial projection methods (cf. Sect. 5).

5. Numerical experiments and applications

In this section, we offer some numerical examples occurring in finite- and infinite-dimensional spaces and applications in optimal control problems to illustrate the computational efficiency of the proposed algorithms compared to some known ones in the literature [29, 40, 38]. All the programs are implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60 GHz computer with RAM 8.00 GB.

5.1. **Theoretical examples.**

Example 5.1. Let the operator $A : \mathbb{R}^m \to \mathbb{R}^m$ be defined by $A(x) = Gx + g$, where $G = BB^\mathsf{T} + S + E$, $g \in \mathbb{R}^m$, $B \in \mathbb{R}^{m\times m}$, $S \in \mathbb{R}^{m\times m}$ is skew-symmetric, and $E \in \mathbb{R}^{m\times m}$ is a diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). Let the feasible set C be a box constraint with the form $C = [-2, 5]^m$. It can be check that A is monotone and Lipschitz continuous with constant $L = \|G\|$. In this example, all entries of B, S are generated randomly in $[-2, 2]$ and E is generated randomly in $[0, 2]$. Let $g = \mathbf{0}$. Then the solution set of the (VIP) is $x^* = \{\mathbf{0}\}$. We use $D_n = \|x_n - x^*\|$ to measure the n-th iteration error of the algorithms. The maximum number of iterations of 2000 as a common stopping criterion and the initial values $x_0 = x_1$ are randomly generated by *rand(m,1)* in MATLAB. Next we test the performance of the proposed algorithms under different parameters. Specifically, we consider the following four cases.
Case 1: *Compare* λ_n. Set $\alpha = 0.6$, $\mu = 0.3$, $\lambda_1 = 0.6$ and $\theta = 0.5$ for the proposed Algorithms 3.1–3.3. Take $\beta = 1.5$ for Algorithm 3.1, $\beta = 0.8$ for Algorithm 3.2, and $\beta = 1.0$ and $\gamma = 1.5$ for Algorithm 3.3. We consider the impact of different parameter choices in the step size criterion (S2) on the proposed algorithms. Specifically, we consider the following two cases: (1) setting $q_n = 1 + 1/n$, $\xi_n = 1 + 1/(n + 1)^{1.1}$ and $\zeta_n = 1/(n + 1)^{1.1}$ in (S2) for all the proposed algorithms; (2) setting $q_n = \xi_n = 1$ and $\zeta_n = 0$ in (S2) (i.e., it becomes the step size criterion (S1)) for all the proposed algorithms. The numerical behavior of the proposed algorithms applying two different step size criteria is expressed in Fig. 1.
Case 2: *Compare* β. Set $\alpha = 0.6$, $\theta = 0.5$, $\mu = 0.3$, $q_n = 1 + 1/n$, $\xi_n = 1 + 1/(n+1)^{1.1}$ and $\zeta_n = 1/(n + 1)^{1.1}$ and $\lambda_1 = 0.6$ for the proposed Algorithms 3.1–3.3. Take $\gamma = 1.5$ for Algorithm 3.3. The numerical performance of the proposed algorithms with different parameters β is shown in Fig. 2.
Case 3: *Compare the inertial parameter* α. Set $\theta = 0.5$, $\mu = 0.3$, $q_n = 1 + 1/n$, $\xi_n = 1+1/(n+1)^{1.1}$ and $\zeta_n = 1/(n+1)^{1.1}$ and $\lambda_1 = 0.6$ for the proposed Algorithms 3.1–3.3. Pick $\gamma = 1.5$ for Algorithm 3.3. The numerical behavior of the proposed algorithms with different parameters α is illustrated in Fig. 3.
Case 4: *Compare the relaxation parameter* θ. Set $\alpha = 0.2$, $\mu = 0.3$, $q_n = 1 + 1/n$, $\xi_n = 1+1/(n+1)^{1.1}$ and $\zeta_n = 1/(n+1)^{1.1}$ and $\lambda_1 = 0.6$ for the proposed Algorithms 3.1–3.3. Choose $\gamma = 1.5$ for Algorithm 3.3. The numerical performance of the proposed algorithms with different parameters θ is demonstrated in Fig. 4.

To end this example, we compare the proposed algorithms with some known methods in the literature, which include the Algorithm 3.2 and the Algorithm 4.1 presented by Shehu et al. [40] (shortly, SLMD Alg. 3.2 and SLMD Alg. 4.1), the Algorithm 2

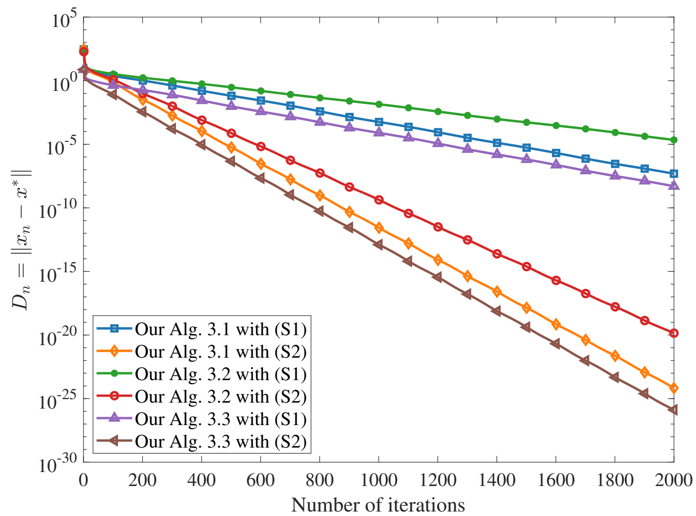
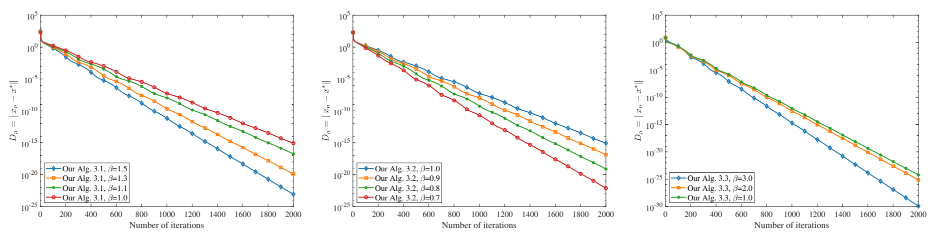
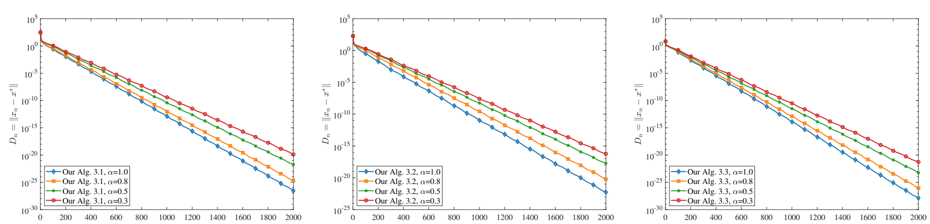


FIGURE 1. The behavior of our algorithms with different stepsize in Example 5.1 ($m = 20$)



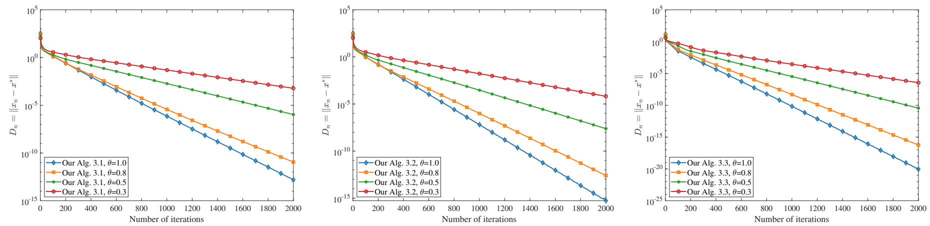
(a) Our Alg. 3.1 (b) Our Alg. 3.2 (c) Our Alg. 3.3

FIGURE 2. Our algorithms with different β in Example 5.1 ($m = 20$)



(a) Our Alg. 3.1 ($\beta = 1.5$) (b) Our Alg. 3.2 ($\beta = 0.8$) (c) Our Alg. 3.3 ($\beta = 1.0$)

FIGURE 3. Our algorithms with different α in Example 5.1 ($m = 20$)



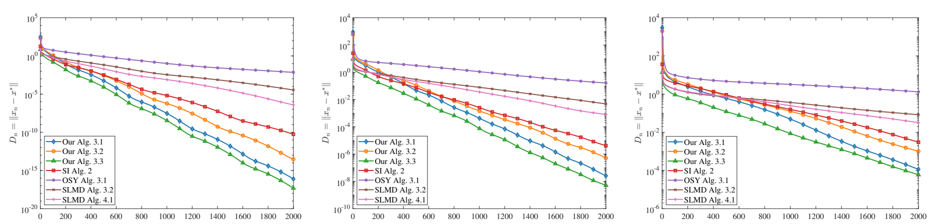
(a) Our Alg. 3.1 ($\beta = 1.0$) (b) Our Alg. 3.2 ($\beta = 0.8$) (c) Our Alg. 3.3 ($\beta = 1.0$)

FIGURE 4. Our algorithms with different θ in Example 5.1 ($m = 20$)

suggested by Shehu and Iyiola [38] (shortly, SI Alg. 2), and the Algorithm 3.1 offered by Ogbuisi, Shehu and Yao [29] (shortly, OSY Alg. 3.1). The parameters of all algorithms are set in Table 1, where "–" in Table 1 indicates that the parameter is not defined in the algorithm. The numerical behavior and numerical results of all algorithms in three different dimensions are shown in Fig. 5 and Table 2, respectively.

TABLE 1. Parameter settings for all algorithms in Example 5.1

Algorithms	α	θ	β	γ	μ	q_n	ξ_n	ζ_n	λ_1
Our Alg. 3.1	1.0	0.4	1.5	–	0.3	$1+\frac{1}{n}$	$1+\frac{1}{(n+1)^{1.1}}$	$\frac{1}{(n+1)^{1.1}}$	0.6
Our Alg. 3.2	1.0	0.4	0.8	–	0.3	$1+\frac{1}{n}$	$1+\frac{1}{(n+1)^{1.1}}$	$\frac{1}{(n+1)^{1.1}}$	0.6
Our Alg. 3.3	1.0	0.4	1.0	1.5	0.3	$1+\frac{1}{n}$	$1+\frac{1}{(n+1)^{1.1}}$	$\frac{1}{(n+1)^{1.1}}$	0.6
SI Alg. 2	0.2	–	–	1.5	0.3	–	–	–	0.6
OSY Alg. 3.1	0.2	0.4	–	–	0.3	–	–	–	0.6
SLMD Alg. 3.2	0.2	0.4	–	1.5	0.3	–	–	–	0.6
SLMD Alg. 4.1	1.0	0.4	–	1.5	0.3	–	–	–	0.6



(a) $m = 20$ (b) $m = 50$ (c) $m = 100$

FIGURE 5. The behavior of all algorithms in different dimensions for Example 5.1

TABLE 2. Numerical results of all algorithms for Example 5.1

Algorithms	$m = 20$		$m = 50$		$m = 100$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	7.83E-17	0.1420	2.60E-08	0.1455	1.16E-04	0.2147
Our Alg. 3.2	3.01E-14	0.1465	5.11E-07	0.1548	1.02E-03	0.2122
Our Alg. 3.3	4.74E-18	0.1716	5.15E-09	0.1732	5.98E-05	0.2112
SI Alg. 2	5.53E-11	0.1400	4.17E-06	0.1581	3.04E-03	0.2063
OSY Alg. 3.1	7.03E-03	0.1259	1.64E-01	0.1465	1.32E+00	0.2080
SLMD Alg. 3.2	3.48E-05	0.1447	4.80E-03	0.1490	8.34E-02	0.2267
SLMD Alg. 4.1	3.89E-07	0.1502	8.09E-04	0.1557	3.21E-02	0.2388

Example 5.2. We consider an example in the Hilbert space $\mathcal{H} = L^2([0,1])$ associated with inner product

$$\langle x, y\rangle := \int_0^1 x(t)y(t)\mathrm{d}t,\ \forall x, y \in \mathcal{H},$$

and induced norm

$$\|x\| := \left(\int_0^1 |x(t)|^2\mathrm{d}t\right)^{1/2},\ \forall x \in \mathcal{H}.$$

Let the feasible set be the unit ball $C := \{x \in \mathcal{H} : \|x\| \le 1\}$. Define an operator $A : C \to \mathcal{H}$ by

$$(Ax)(t) = \int_0^1 \left[x(t) - G(t,s)g(x(s))\right]\mathrm{d}s + h(t),\ t \in [0,1],\ x \in C,$$

where

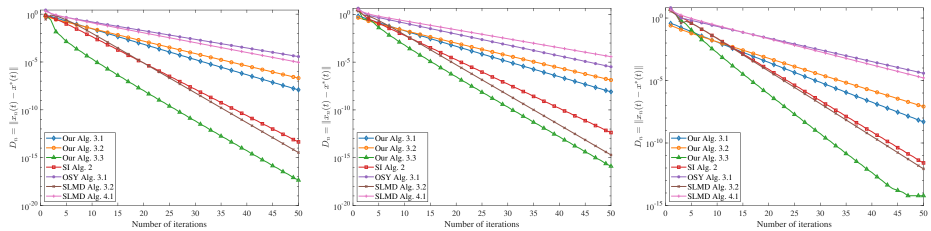
$$G(t,s) = \frac{2ts\mathrm{e}^{t+s}}{e\sqrt{\mathrm{e}^2 - 1}}, \quad g(x) = \cos x, \quad h(t) = \frac{2t\mathrm{e}^t}{e\sqrt{\mathrm{e}^2 - 1}}.$$

It is known that A is monotone and L-Lipschitz continuous with $L = 2$ (see [18]), and $x^*(t) = 0$ is the solution of the corresponding variational inequality problem. We also compare the proposed algorithms with the ones mentioned in Example 5.1. The parameters of all algorithms are set in Table 3.

TABLE 3. Parameter settings for all algorithms in Example 5.2

Algorithms	α	θ	β	γ	μ	q_n	ξ_n	ζ_n	λ_1
Our Alg. 3.1	0.2	1.0	1.3	–	0.3	$1+\frac{1}{n}$	$1+\frac{1}{(n+1)^{1.1}}$	$\frac{1}{(n+1)^{1.1}}$	0.6
Our Alg. 3.2	0.2	1.0	0.8	–	0.3	$1+\frac{1}{n}$	$1+\frac{1}{(n+1)^{1.1}}$	$\frac{1}{(n+1)^{1.1}}$	0.6
Our Alg. 3.3	0.2	1.0	1.0	1.5	0.3	$1+\frac{1}{n}$	$1+\frac{1}{(n+1)^{1.1}}$	$\frac{1}{(n+1)^{1.1}}$	0.6
SI Alg. 2	0.2	–	–	1.5	0.3	–	–	–	0.6
OSY Alg. 3.1	0.2	1.0	–	–	0.3	–	–	–	0.6
SLMD Alg. 3.2	0.2	0.9	–	1.5	0.3	–	–	–	0.6
SLMD Alg. 4.1	1.0	0.4	–	1.5	0.3	–	–	–	0.6

We use $D_n = \|x_n(t) - x^*(t)\|$ to measure the n-th iteration error of all algorithms and choose the maximum number of iterations of 50 as the common stopping criterion. Figure 6 and Table 4 show the numerical performance and numerical results of all algorithms for three different types of initial values $x_0(t) = x_1(t)$.



(a) $x_0(t) = x_1(t) = 10t^3$ (b) $x_0(t) = x_1(t) = 10\sin(6t)$ (c) $x_0(t) = x_1(t) = 10\log(4t)$

FIGURE 6. The behavior of all algorithms with different initial values for Example 5.2

TABLE 4. Numerical results of all algorithms with different initial values for Example 5.2

Algorithms	$x_1(t) = 10t^3$		$x_1(t) = 10\sin(6t)$		$x_1(t) = 10\log(4t)$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	1.23E-08	22.9275	7.90E-09	22.3638	4.96E-09	23.5230
Our Alg. 3.2	2.09E-07	21.7844	1.35E-07	21.3949	8.29E-08	22.1280
Our Alg. 3.3	4.45E-18	23.6675	1.29E-16	24.1700	6.04E-15	25.7202
SI Alg. 2	4.43E-14	23.6249	4.30E-13	24.8159	2.55E-12	25.7061
OSY Alg. 3.1	3.68E-05	20.8452	3.28E-06	21.2197	3.83E-05	23.0713
SLMD Alg. 3.2	3.39E-15	25.7996	1.94E-15	24.5903	8.49E-13	25.7472
SLMD Alg. 4.1	9.23E-06	24.0537	3.71E-05	24.5627	1.62E-05	25.1763

Example 5.3. Let $\mathcal{H} = L^2([0,1])$ be an infinite-dimensional Hilbert space with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\,\mathrm{d}t,\ \forall x, y \in \mathcal{H}$$

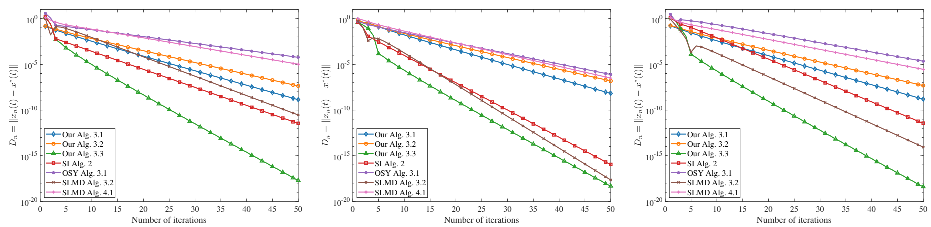
and induced norm

$$\|x\| = \left(\int_0^1 |x(t)|^2\,\mathrm{d}t\right)^{1/2},\ \forall x \in \mathcal{H}.$$

Assume that r and R are two positive real numbers such that $R/(k+1) < r/k < r < R$ for some $k > 1$. Let the feasible set be defined by $C = \{x \in \mathcal{H} : \|x\| \le r\}$ and the operator $A : \mathcal{H} \to \mathcal{H}$ be given by

$$Ax = (R - \|x\|)x, \quad \forall x \in \mathcal{H}\,.$$

Note that A is Lipschitz continuous and pseudo-monotone rather than monotone (see [42, Example 4.2]). For the experiment, we choose $R = 1.5$, $r = 1$, $k = 1.1$. The solution of the variational inequality problem (VIP) with A and C given above is $x^*(t) = 0$. The parameters of all algorithms are set as in Table 3. The maximum number of iterations of 50 is used as a common stopping criterion and $D_n = \|x_n(t) - x^*(t)\|$ is used to measure the error of the n-th iteration step of all algorithms. The numerical performance and numerical results of all algorithms with three different initial values $x_0(t) = x_1(t)$ are stated in Fig. 7 and Table 5.



(a) $x_0(t) = x_1(t) = 9t^3$ (b) $x_0(t) = x_1(t) = \log(t)$ (c) $x_0(t) = x_1(t) = 6^t$

FIGURE 7. The behavior of all algorithms with different initial values for Example 5.3

TABLE 5. Numerical results of all algorithms with different initial values for Example 5.3

Algorithms	$x_1(t) = 9t^3$		$x_1(t) = \log(t)$		$x_1(t) = 6^t$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	1.38E-09	16.8353	6.69E-09	16.8393	1.57E-09	16.7370
Our Alg. 3.2	4.37E-08	19.1554	1.54E-07	16.3324	4.86E-08	16.1349
Our Alg. 3.3	2.03E-18	21.9592	4.75E-19	19.1315	4.01E-19	19.1473
SI Alg. 2	3.48E-12	21.4797	1.11E-16	19.0594	3.62E-12	19.0252
OSY Alg. 3.1	5.94E-05	16.8186	7.88E-07	15.8354	2.22E-05	15.8098
SLMD Alg. 3.2	2.89E-11	18.8474	2.22E-18	18.8823	8.79E-15	18.8880
SLMD Alg. 4.1	1.05E-05	18.8571	3.15E-07	19.0200	2.85E-06	18.9539

Remark 5.1. We have the following observations for Examples 5.1–5.3.

(1) It can be seen from Fig. 1 that our algorithms using the non-monotonic step size criterion (S2) have a faster convergence speed and higher accuracy than our algorithms using the non-increasing step size criterion (S1). This shows that the non-monotonic step size rule introduced in this paper is useful and efficient.

(2) As can be seen in Figs. 2, 3, and 4, different parameters β, α, and θ have different effects on the proposed algorithms. Specifically, we have the following observations: (1) the proposed Algorithms 3.1 and 3.3 have a higher accuracy

when the parameter β is larger, while the proposed Algorithm 3.2 has a better performance when the parameter β is smaller (cf. Fig. 2); (2) our three algorithms perform better and better as the inertial factor α increases (cf. Fig. 3); and (3) our three algorithms have a higher accuracy when the relaxation factor θ becomes larger (cf. Fig. 4).

(3) The performance of our algorithms is better than the schemes presented in [29, 40, 38]. More precisely, our algorithms have a higher accuracy and faster convergence speed than the ones in [29, 40, 38] under the same stopping conditions achieved, and these results are not related to the size of the dimension and the choice of initial values. Moreover, it can be seen from Tables 2, 4 and 5 that the computational complexity of our algorithms is the same as that of the schemes in [29, 40, 38], i.e., the time consumed by these algorithms is not much different. Thus, the methods proposed in this paper are efficient and robust.

(4) Notice that the solutions x^* of the three examples provided are known, and we all use $D_n = \|x_n - x^*\|$ to denote the n-th iteration error for all algorithms. It can be seen intuitively from Figs. 1, 2, 3, 4, 5, 6 and 7 that the sequence $\{\|x_n - x^*\|\}$ generated by our three algorithms is monotonically decreasing, which further implies that the even subsequence $\{\|x_{2n} - x^*\|\}$ is also monotonically decreasing (i.e., $\|x_{2n+2} - x^*\| \leq \|x_{2n} - x^*\|$), and this observation also verifies the fact in Lemma 3.2, Lemma 3.4 and Lemma 3.6 that $\{x_{2n}\}$ is Fejér monotone with respect to $\mathrm{VI}(C, A)$, the solution set of the variational inequality problem (VIP).

(5) It should be noted that the operator A in Example 5.3 is pseudo-monotone rather than monotone, which means that the methods proposed in the literature (see, e.g., [13, 14, 12, 16, 15, 37, 39]) for solving monotone variational inequalities will not be available. On the other hand, many of the fixed-step methods introduced in the literature (see, e.g., [14, 12, 34]) will also fail in Example 5.3 because the Lipschitz constant of the operator A in Example 5.3 is unknown.

Therefore, our three algorithms proposed in this paper are efficient, useful and have a broader scope of applications.

5.2. **Application to optimal control problems.** In this subsection, we use the proposed algorithms to solve the (VIP) that appears in optimal control problems. Assume that $L_2([0,T],\mathbb{R}^m)$ represents the square-integrable Hilbert space with inner product

$$\langle p, q\rangle = \int_0^T \langle p(t), q(t)\rangle \, \mathrm{d}t$$

and norm $\|p\| = \sqrt{\langle p, p\rangle}$. The optimal control problem is described as follows:

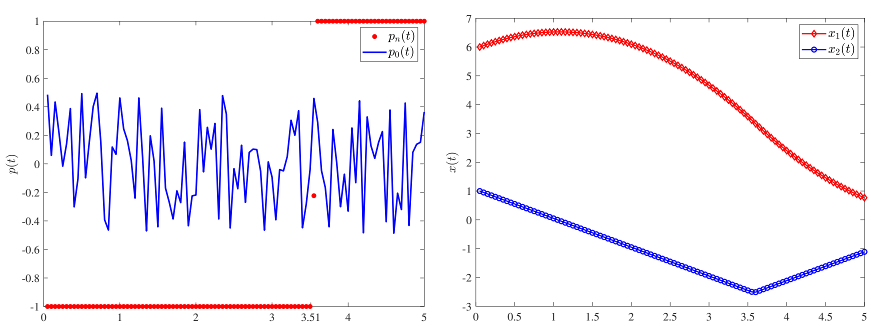
$$\begin{cases} p^*(t) \in \operatorname{Argmin}\{g(p) \mid p \in V\}, \\ g(p) = \Phi(x(T)), \\ V = \left\{p(t) \in L_2([0,T],\mathbb{R}^m) : p_i(t) \in \left[p_i^-, p_i^+\right], i = 1, 2, \ldots, m\right\}, \\ \text{s.t. } \dot{x}(t) = Q(t)x(t) + W(t)p(t), \quad 0 \leq t \leq T, \quad x(0) = x_0, \end{cases} \tag{5.1}$$

where $g(p)$ means the terminal objective function, Φ is a convex and differentiable defined on the attainability set, $p(t)$ denotes the control function, V represents a set of feasible controls composed of m piecewise continuous functions, $x(t)$ stands for the trajectory, and $Q(t) \in \mathbb{R}^{n\times n}$ and $W(t) \in \mathbb{R}^{n\times m}$ are given continuous matrices for every $t \in [0, T]$. By the solution of problem (5.1), we mean a control $p^*(t)$ and a corresponding (optimal) trajectory $x^*(t)$ such that its terminal value $x^*(T)$ minimizes objective function $g(p)$. It is known that the optimal control problem (5.1) can be transformed into a variational inequality problem (see [33, 44]). We first use the classical Euler discretization method to decompose the optimal control problem (5.1) and then apply the proposed algorithms to solve the variational inequality problem corresponding to the discretized version of the problem (see [33, 44] for more details).

Example 5.4 (Rocket car [33])**.**

$$
\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\left((x_1(5))^2 + (x_2(5))^2\right), \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\
& \dot{x}_2(t) = p(t), \quad \forall t \in [0, 5], \\
& x_1(0) = 6, \quad x_2(0) = 1, \quad p(t) \in [-1, 1].
\end{aligned}
$$

The exact optimal control of Example 5.4 is $p^*(t) = -1$ if $t \in (0, 3.517]$ and $p^*(t) = 1$ if $t \in (3.517, 5]$. We compare the proposed algorithms with the ones mentioned in Example 5.1. The parameters of all algorithms are set as in Table 3. The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$ and the stopping criterion is either $E_n = \|p_{n+1} - p_n\| \leq 10^{-4}$ or the maximum number of iterations is reached 1000. The approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.1 are plotted in Fig. 8.



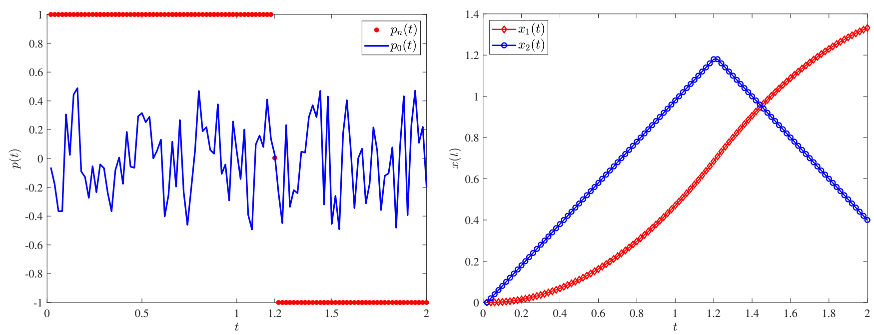
(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 8. Numerical results for Example 5.4

Example 5.5 (See [3])**.**

$$\begin{aligned} \text{minimize} \quad & -x_1(2)+(x_2(2))^2, \\ \text{subject to} \quad & \dot{x}_1(t)=x_2(t), \\ & \dot{x}_2(t)=p(t), \quad \forall t\in[0,2], \\ & x_1(0)=0, \quad x_2(0)=0, \quad p(t)\in[-1,1]. \end{aligned}$$

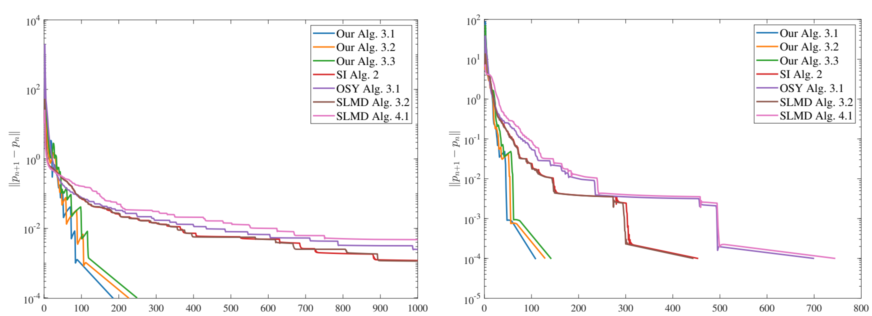
The exact optimal control of Example 5.5 is $p^*(t)=1$ if $t\in[0,1.2)$ and $p^*(t)=-1$ if $t\in(1.2,2]$. The parameters and stopping criteria of all algorithms are the same as in Example 5.4. Fig. 9 gives the approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.2.



(a) Initial and optimal controls (b) Optimal trajectories

FIGURE 9. Numerical results for Example 5.5

Finally, the numerical results of all algorithms in Examples 5.4 and 5.5 are shown in Fig. 10 and Table 6.



(a) Example 5.4 (b) Example 5.5

FIGURE 10. Numerical behavior of all algorithms in Examples 5.4 and 5.5

TABLE 6. Numerical results of all algorithms in Examples 5.4 and 5.5

Algotithms	Example 5.4			Example 5.5		
	Iter.	CPU (s)	E_n	Iter.	CPU (s)	E_n
Our Alg. 3.1	185	0.0765	9.7251E-05	109	0.0501	9.5801E-05
Our Alg. 3.2	227	0.0813	9.8148E-05	129	0.0471	9.9337E-05
Our Alg. 3.3	249	0.0866	9.9584E-05	142	0.0501	9.7687E-05
SI Alg. 2	1000	0.3181	1.1931E-03	454	0.1431	9.9589E-05
OSY Alg. 3.1	1000	0.3118	2.4855E-03	700	0.2121	9.9927E-05
SLMD Alg. 3.2	1000	0.3188	1.1461E-03	444	0.1353	9.9757E-05
SLMD Alg. 4.1	1000	0.3053	4.7566E-03	745	0.2246	9.9958E-05

Remark 5.2. From Fig. 8, Fig. 9, Fig. 10 and Table 6, it is known that the three algorithms proposed in this paper can be applied to solve optimal control problems. Moreover, they perform better than the schemes presented in the literature [29, 40, 38]. Specifically, they require fewer iterations and less execution time than the algorithms in [29, 40, 38] to achieve the same stopping conditions, and these results are independent of the choice of the problem. Thus, our algorithms are efficient and robust.

6. Conclusions

In this paper, three new single projection methods with alternating inertial extrapolation steps and relaxation effects are introduced to solve pseudo-monotone variational inequality problems in real Hilbert spaces. The proposed algorithms are inspired by the alternated inertial method, the subgradient extragradient method, the projection and contraction method, and the relaxation method. Our schemes apply a non-monotonic step size criterion allowing them to work without the prior knowledge of the Lipschitz constant of the operator. The weak convergence of the iterative sequences generated by the proposed algorithms is proved under some suitable conditions. The Fejér monotonicity of the even subsequences generated by our algorithms is recovered and the linear convergence rate is confirmed under the assumption that the operator is strongly pseudo-monotone. Finally, some numerical tests and applications are given to illustrate the performance of the presented approaches compared to some known inertial projection methods which also include some recent alternated inertial methods.

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