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REVISITING PROJECTION AND CONTRACTION ALGORITHMS FOR SOLVING VARIATIONAL INEQUALITIES AND APPLICATIONS

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Abstract. We provide two novel projection and contraction algorithms to find the minimum-norm solution of the variational inequality problem with a pseudomonotone and non-Lipschitz continuous operator in a real Hilbert space. Our algorithms can work adaptively without requiring the prior information of the Lipschitz constant of the operator. Strong convergence theorems for the suggested iterative algorithms are established under suitable conditions. Some numerical experiments are discussed to demonstrate the computational efficiency of the proposed algorithms in comparison with several existing ones.
Keywords. Projection and contraction method; Subgradient extragradient method; Pseudomonotone mapping; Uniformly continuous; Variational inequality.

1. INTRODUCTION

The purpose of this paper is to develop two adaptive iterative algorithms to find the solutions of variational inequality problems in infinite-dimensional Hilbert spaces. Let $\mathscr{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and let $Q:\mathscr{H}\to\mathscr{H}$ be a nonlinear operator. Recall that the variational inequality problem (shortly, VIP) is formed as follows.

$$\text{Find } x^\dagger\in\mathscr{C} \text{ such that } \langle Qx^\dagger, x-x^\dagger\rangle\geq 0,\quad \forall x\in\mathscr{C}. \tag{VIP}$$

We denote Ω as the solution set of (VIP) and assume it to be always nonempty in this paper. The variational inequality can be used as a model for solving many practical problems (such as optimal control problems, image processing problems, signal recovery problems and so on); see, e.g., [1, 2, 3]. In the last few decades, many efficient solution methods were proposed for solving variational inequality problems; see, e.g., [4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein.

In this paper, we concentrate on projection-based methods. Korpelevich [4] introduced a two-step iterative scheme (now known as the extragradient method) to solve the saddle point problem, which was subsequently extended to solve the variational inequality problem. The

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extragradient method requires that two projections on the feasible set need to be computed in each iteration. More precisely, the extragradient method is described as follows:

$$\begin{cases} d_n = Proj_{\mathscr{C}}(x_n - \chi Q x_n), \\ x_{n+1} = Proj_{\mathscr{C}}(x_n - \chi Q d_n), \end{cases} \tag{EGM}$$

where step size χ is limited by the Lipschitz of mapping Q. Under some suitable conditions, the weak convergence of the iterative sequence generated by the (EGM) is verified in real Hilbert spaces. Notice that (EGM) needs to evaluate the projection on the feasible set twice in each iteration. It is known that computing the projection is not easy especially when the feasible set is complex. To reduce the projection computation on the feasible set for each iteration, some improved versions of (EGM) were developed to solve the variational inequality problem (see, e.g., [6, 7, 8]). We present here two known methods that require the the computation of the projection on the feasible set only once in each iteration. The first is the Tseng extragradient method proposed by Tseng [7], which is stated as follows:

$$\begin{cases} d_n = Proj_{\mathscr{C}}(x_n - \chi Q x_n), \\ x_{n+1} = d_n - \chi (Q d_n - Q x_n), \end{cases} \tag{TEGM}$$

where the range of step size χ is related to the Lipschitz constant of mapping Q. Tseng replaced the projection calculation on the feasible set in the (EGM) with a display calculation in the second step. The weak convergence of (TEGM) was proved under some suitable conditions in Hilbert spaces. The second is the subgradient extragradient method introduced by Censor et al. [8]. The iterative process of this method is shown as follows:

$$\begin{cases} d_n = Proj_{\mathscr{C}}(x_n - \chi Q x_n), \\ T_n = \{x \in \mathscr{H} : \langle x_n - \chi Q x_n - d_n, x - d_n \rangle \leq 0\}, \\ x_{n+1} = Proj_{T_n}(x_n - \chi Q d_n), \end{cases} \tag{SEGM}$$

where the value of step size χ is determined with respect to the Lipschitz constant of mapping Q. It is noted that the projection on the half-space T_n can be computed explicitly (see, e.g., [14]). As a result, (SEGM) only needs to compute the projection on the feasible set once each iteration, and its weak convergence in Hilbert spaces can also be achieved.

Notice that the methods proposed in [6, 7, 8, 9, 11, 12] obtained weak convergence in an infinite-dimensional Hilbert space. It is known that strong convergence is preferable to weak convergence in infinite-dimensional spaces. Recently, Thong and Gibali [15] introduced two new algorithms, which were inspired by the modified subgradient extragradient method suggested by [16], the Mann-type method, and the viscosity-type method, to solve the monotone variational inequality problem. More precisely, their iterative schemes are expressed as follows:

Simply changing the third step in Algorithm 1.1 to $x_{n+1} = \tau_n f(x_n) + (1 - \tau_n) z_n$ (where f is a contraction mapping) yields the Algorithm 3.2 proposed by Thong and Gibali [15]. Under some suitable conditions, the algorithms proposed by Thong and Gibali [15] are strongly convergent in real Hilbert spaces. In addition, Gibali et al. [17] provided two simple projection-type methods to find the solutions of the monotone variational inequality problem in real Hilbert spaces. The forms of their algorithms are as follows:

Replace the third step in Algorithm 1.2 with $x_{n+1} = \tau_n f(x_n) + (1 - \tau_n) z_n$ to obtain the Algorithm 3.2 proposed by Gibali et al. [17]. The strong convergence of the iterative sequences generated by the two algorithms proposed by Gibali et al. [17] was established under mild

Algorithm 1.1 The Algorithm 3.1 of Thong and Gibali [15]

Initialization: Take $\rho > 0$, $\ell \in (0,1)$, $\eta \in (0,1)$, and $\psi \in (0,2)$. Let $x_0 \in \mathscr{H}$.
Iterative Steps: Given the iterate $x_n\,(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $d_n = Proj_{\mathscr{C}}(x_n - \chi_n Q x_n)$, where the step size χ_n is chosen to be the largest $\chi \in \{\rho, \rho\ell, \rho\ell^2, \ldots\}$ satisfying

$$\chi \|Qx_n - Qd_n\| \leq \eta \|x_n - d_n\|. \tag{1.1}$$

If $x_n = d_n$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 2**.
Step 2. Compute $z_n = Proj_{T_n}(x_n - \psi \chi_n \sigma_n Q d_n)$, where

$$T_n = \{x \in \mathscr{H} \mid \langle x_n - \chi_n Q x_n - d_n, x - d_n \rangle \leq 0\},$$

and

$$\sigma_n = (1-\eta)\frac{\|x_n - d_n\|^2}{\|\delta_n\|^2}, \quad \delta_n = x_n - d_n - \chi_n (Qx_n - Qd_n). \tag{1.2}$$

Step 3. Compute $x_{n+1} = (1 - \tau_n - \sigma_n)x_n + \sigma_n z_n$.
Set $n = n+1$ and go to **Step 1**.

Algorithm 1.2 The Algorithm 3.1 of Gibali et al. [17]

Initialization: Take $\rho > 0$, $\ell \in (0,1)$, $\eta \in (0,1)$, and $\psi \in (0,2)$. Let $x_0 \in \mathscr{H}$.
Iterative Steps: Given the iterate $x_n\,(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $d_n = Proj_{\mathscr{C}}(x_n - \chi_n Q x_n)$, where the step size χ_n is defined in (1.1). If $x_n = d_n$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 2**.
Step 2. Compute $z_n = x_n - \psi \sigma_n \delta_n$, where σ_n and δ_n are defined in (1.2).
Step 3. Compute $x_{n+1} = (1 - \tau_n - \sigma_n)x_n + \sigma_n z_n$.
Set $n = n+1$ and go to **Step 1**.

conditions. Notice that the algorithms introduced in [15, 16, 17] were designed to solve monotone variational inequality problems. Recently, the class of pseudomonotone mappings as a broader class of mappings than the class of monotone mappings attracted a great deal of research interest from the scholars in the optimization community, and a lot of numerical methods to solve pseudomonotone variational inequality problems were introduced and investigated; see, e.g., [12, 13, 18, 19, 20, 21, 22, 23] and the references therein. On the other hand, many researchers investigated the concept of inertial as a way to accelerate the convergence speed of algorithms. The next iteration of an inertial-type method is determined by the combination of the previous two (or more) iterations. It is worth noting that this minor adjustment can enhance the convergence speed of the algorithms that do not use inertial. Recently, a large number of iterative algorithms were proposed to solve variational inequality problems, equilibrium problems, fixed point problems, splitting problems, and other optimization problems; see, e.g., [11, 19, 21, 24] and the references therein.

Inspired and motivated by results above, we introduce in this paper two novel inertial extragradient methods to solve the pseudomonotone variational inequality problem in a real Hilbert space. Our contributions in this paper are as follows.

(1) The suggested algorithms are proved to be strongly convergent in an infinite-dimensional Hilbert space, which is preferable to the weak convergence results that already exist in the literature [9, 11, 12].
(2) Our two algorithms can work without known the Lipschitz constant of the mapping involved, which improves the fixed-step methods presented in [9, 10, 11, 12, 13, 19].
(3) The suggested iterative algorithms can solve pseudomonotone and non-Lipschitz continuous variational inequalities, which extend the methods used in the literature (see, e.g., [9, 10, 11, 15, 16, 17]) for solving monotone and Lipschitz continuous (or non-Lipschitz continuous) variational inequalities and the results used in the literature (see, e.g., [18, 19, 20, 21]) for solving pseudomonotone and Lipschitz continuous variational inequalities.
(4) The proposed algorithms incorporate inertial terms and use two different step sizes in each iteration, which enhances the convergence speed and the accuracy of the algorithms in the literature [15, 17, 22, 23].

The rest of this paper is organized as follows. Some basic preliminaries are collected in Section 2. Section 3 aims to present two adaptive inertial projection and contraction algorithms and analyze their convergence. The computational efficiency of the suggested algorithms compared to some known schemes is explained in detail in Section 4. Finally, a brief summary of the paper is given in Section 5, the last section.

2. Preliminaries

Let $\mathscr{C}$ be a nonempty, closed, and convex subset of a real Hilbert space $\mathscr{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Recall that an operator $Q : \mathscr{H} \to \mathscr{H}$ is said to be:

(1) *L-Lipschitz continuous* with $L > 0$ if $\|Qx - Qy\| \le L\|x - y\|$, $\forall x, y \in \mathscr{H}$;
(2) *monotone* if $\langle Qx - Qy, x - y\rangle \ge 0$, $\forall x, y \in \mathscr{H}$;
(3) *pseudomonotone* if $\langle Qx, y - x\rangle \ge 0 \Rightarrow \langle Qy, y - x\rangle \ge 0$, $\forall x, y \in \mathscr{H}$;
(4) *sequentially weakly continuous* if each sequence $\{x_n\}$ converges weakly to x implies that $\{Qx_n\}$ converges weakly to Qx.

Recall that the *metric projection*, $Proj_{\mathscr{C}}$, of $\mathscr{H}$ onto $\mathscr{C}$, is defined by

$$Proj_{\mathscr{C}}(x) = \arg\min\{\|x - y\|, y \in \mathscr{C}\}.$$

It is known that $Proj_{\mathscr{C}}$ has the following basic properties:

$$\|Proj_{\mathscr{C}}(x) - Proj_{\mathscr{C}}(y)\|^2 \le \langle Proj_{\mathscr{C}}(x) - Proj_{\mathscr{C}}(y), x - y\rangle, \forall x, y \in \mathscr{H} \tag{2.1}$$

and

$$\langle x - Proj_{\mathscr{C}}(x), y - Proj_{\mathscr{C}}(x)\rangle \le 0, \forall x \in \mathscr{H}, \forall y \in \mathscr{C}, \tag{2.2}$$

We give some explicit formulas to calculate projections on special feasible sets.

(1) The projection of x onto a half-space $H_{u,v} = \{x : \langle u, x\rangle \le v\}$ is given by

$$Proj_{H_{u,v}}(x) = x - \max\left\{\frac{\langle u, x\rangle - v}{\|u\|^2}, 0\right\} u.$$

(2) The projection of x onto a box $\text{Box}[a, b] = \{x : a \le x \le b\}$ is given by

$$Proj_{\text{Box}[a,b]}(x)_i = \min\{b_i, \max\{x_i, a_i\}\}.$$

(3) The projection of x onto a ball $B[p,q] = \{x : \|x-p\| \leq q\}$ is given by

$$Proj_{B[p,q]}(x) = p + \frac{q}{\max\{\|x-p\|, q\}}(x-p).$$

The following lemma is essential for the convergence analysis of our main results.

Lemma 2.1 ([25]). *Let $\{x_n\}$ be a positive sequence, $\{u_n\}$ be a sequence of real numbers and $\{\tau_n\}$ be a sequence in $(0,1)$ such that $\sum_{n=1}^{\infty} \tau_n = \infty$. Assume that*

$$x_{n+1} \leq (1-\tau_n)x_n + \tau_n u_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} u_{n_k} \leq 0$ for any subsequence $\{x_{n_k}\}$ of $\{x_n\}$ satisfying

$$\liminf_{k\to\infty} \left(x_{n_k+1} - x_{n_k}\right) \geq 0,$$

then $\lim_{n\to\infty} x_n = 0$.

3. Main Results

In this section, we present two new iterative algorithms to solve pseudomonotone variational inequalities in infinite-dimensional Hilbert spaces. Our algorithms work well without the prior knowledge of the Lipschitz constant of the operator, and they can be used to solve non-Lipschitz continuous variational inequalities. Before starting to present our algorithms, we first assume that the following conditions are satisfied for our algorithms.

(C1) The feasible set $\mathscr{C}$ is a nonempty, closed, and convex subset of $\mathscr{H}$, and the solution set of (VIP) is nonempty, that is, $\Omega \neq \emptyset$;

(C2) The operator $Q : \mathscr{H} \to \mathscr{H}$ is pseudomonotone, uniformly continuous on $\mathscr{H}$, and sequentially weakly continuous on $\mathscr{C}$.

(C3) Let $\{\zeta_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\zeta_n}{\tau_n} = 0$, where $\{\tau_n\} \subset (0,1)$ satisfies $\lim_{n\to\infty} \tau_n = 0$ and $\sum_{n=1}^{\infty} \tau_n = \infty$.

Now, we are in a position to state our Algorithm 3.1.

We can obtain the following conclusions of Lemmas 3.1 and 3.2 by a simple modification of Lemmas 3.1 and 3.3 in [26], respectively. To avoid repetitive expressions, we omit their proofs.

Lemma 3.1 ([26]). *Assume Condition (C2) holds. The Armijo-like criteria (3.2) is well defined.*

Lemma 3.2 ([26]). *Assume Condition (C2) holds. Let $\{u_n\}$ and $\{d_n\}$ be two sequences formed by Algorithm 3.1. If there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\{u_{n_k}\}$ converges weakly to $z \in \mathscr{H}$ and $\lim_{k\to\infty} \|u_{n_k} - d_{n_k}\| = 0$, then $z \in \Omega$.*

Lemma 3.3. *Assume Condition (C2) holds. Let sequences $\{u_n\}$, $\{d_n\}$, and $\{x_{n+1}\}$ be formed by Algorithm 3.1. Then*

$$\begin{aligned}\|x_{n+1} - x^\dagger\|^2 \leq \|u_n - x^\dagger\|^2 - \|u_n - x_{n+1} - \frac{\psi}{\beta}\gamma_n\delta_n\|^2 \\ - \frac{\psi}{\beta^2}(2\beta - \psi)\frac{(1-\beta\eta)^2}{(1+\beta\eta)^2}\|u_n - d_n\|^2, \ \forall x^\dagger \in \Omega.\end{aligned}$$

Algorithm 3.1 A novel modified inertial subgradient extragradient method

Initialization: Take $\alpha > 0$, $\rho > 0$, $\ell \in (0,1)$, $\eta \in (0,1)$, $\psi \in (0,2/\eta)$, and $\beta \in (\psi/2, 1/\eta)$. Select sequences $\{\zeta_n\}$ and $\{\tau_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathscr{H}$.

Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:

Step 1. Compute $u_n = (1-\tau_n)(x_n + \alpha_n(x_n - x_{n-1}))$, where

$$\alpha_n = \begin{cases} \min\left\{\dfrac{\zeta_n}{\|x_n - x_{n-1}\|}, \alpha\right\}, & \text{if } x_n \neq x_{n-1}; \\ \alpha, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $d_n = Proj_{\mathscr{C}}(u_n - \beta\chi_n Q u_n)$, where χ_n is chosen to be the largest $\chi \in \{\rho, \rho\ell, \rho\ell^2, \ldots\}$ satisfying

$$\chi\|Qu_n - Qd_n\| \leq \eta\|u_n - d_n\|. \tag{3.2}$$

If $u_n = d_n$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 3**.

Step 3. Compute $x_{n+1} = Proj_{H_n}(u_n - \psi\chi_n\gamma_n Q d_n)$, where

$$H_n = \{x \in \mathscr{H} \mid \langle u_n - \beta\chi_n Q u_n - d_n, x - d_n\rangle \leq 0\},$$

and

$$\gamma_n = (1-\beta\eta)\frac{\|u_n - d_n\|^2}{\|\delta_n\|^2}, \quad \delta_n = u_n - d_n - \beta\chi_n(Qu_n - Qd_n). \tag{3.3}$$

Set $n = n+1$ and go to **Step 1**.

Proof. In view of (2.1) and $x^\dagger \in \Omega \subset \mathscr{C} \subset H_n$, we obtain

$$\begin{aligned}
&2\|x_{n+1} - x^\dagger\|^2 \\
&= 2\|Proj_{H_n}(u_n - \psi\chi_n\gamma_n Qd_n) - Proj_{H_n}(x^\dagger)\|^2 \\
&\leq 2\langle x_{n+1} - x^\dagger, u_n - \psi\chi_n\gamma_n Qd_n - x^\dagger\rangle \\
&= \|x_{n+1} - x^\dagger\|^2 + \|u_n - \psi\chi_n\gamma_n Qd_n - x^\dagger\|^2 - \|x_{n+1} - u_n + \psi\chi_n\gamma_n Qd_n\|^2 \\
&= \|x_{n+1} - x^\dagger\|^2 + \|u_n - x^\dagger\|^2 + \psi^2\chi_n^2\gamma_n^2\|Qd_n\|^2 - 2\langle u_n - x^\dagger, \psi\chi_n\gamma_n Qd_n\rangle \\
&\quad - \|x_{n+1} - u_n\|^2 - \psi^2\chi_n^2\gamma_n^2\|Qd_n\|^2 - 2\langle x_{n+1} - u_n, \psi\chi_n\gamma_n Qd_n\rangle \\
&= \|x_{n+1} - x^\dagger\|^2 + \|u_n - x^\dagger\|^2 - \|x_{n+1} - u_n\|^2 - 2\langle x_{n+1} - x^\dagger, \psi\chi_n\gamma_n Qd_n\rangle,
\end{aligned}$$

which gives that

$$\|x_{n+1} - x^\dagger\|^2 \leq \|u_n - x^\dagger\|^2 - \|x_{n+1} - u_n\|^2 - 2\psi\chi_n\gamma_n\langle x_{n+1} - x^\dagger, Qd_n\rangle. \tag{3.4}$$

From $x^\dagger \in \Omega$ and $d_n \in \mathscr{C}$, one sees that $\langle Qx^\dagger, d_n - x^\dagger\rangle \geq 0$. This combining with the pseudomonotonicity of Q yields that $\langle Qd_n, d_n - x^\dagger\rangle \geq 0$, which is equivalent to

$$\langle Qd_n, x_{n+1} - x^\dagger\rangle \geq \langle Qd_n, x_{n+1} - d_n\rangle.$$

Note that $\psi > 0$, $\chi_n > 0$, and $\gamma_n > 0$. Thus

$$-2\psi\chi_n\gamma_n\langle Qd_n, x_{n+1} - x^\dagger\rangle \leq -2\psi\chi_n\gamma_n\langle Qd_n, x_{n+1} - d_n\rangle. \tag{3.5}$$

According to $x_{n+1} \in H_n$, one obtains $\langle u_n - \beta\chi_n Qu_n - d_n, x_{n+1} - d_n\rangle \leq 0$. This means that

$$\langle u_n - d_n - \beta\chi_n(Qu_n - Qd_n), x_{n+1} - d_n\rangle \leq \beta\chi_n\langle Qd_n, x_{n+1} - d_n\rangle. \tag{3.6}$$

Combining the definition of δ_n, (3.5), and (3.6), we deduce

$$-2\psi\chi_n\gamma_n\langle Qd_n, x_{n+1}-x^\dagger\rangle \le -2\frac{\psi}{\beta}\gamma_n\langle\delta_n, u_n-d_n\rangle + 2\frac{\psi}{\beta}\gamma_n\langle\delta_n, u_n-x_{n+1}\rangle. \tag{3.7}$$

By using the definitions of γ_n and δ_n, and (3.2), we derive

$$\begin{aligned}\langle\delta_n, u_n-d_n\rangle &\ge \|u_n-d_n\|^2-\beta\chi_n\|Qu_n-Qd_n\|\|u_n-d_n\|\\ &\ge \|u_n-d_n\|^2-\beta\eta\|u_n-d_n\|^2\\ &= (1-\beta\eta)\|u_n-d_n\|^2\\ &= \gamma_n\|\delta_n\|^2,\end{aligned}$$

which further yields that

$$-2\frac{\psi}{\beta}\gamma_n\langle\delta_n, u_n-d_n\rangle \le -2\frac{\psi}{\beta}\gamma_n^2\|\delta_n\|^2. \tag{3.8}$$

It follows from the basic inequality $2ab = a^2+b^2-(a-b)^2$ that

$$2\frac{\psi}{\beta}\gamma_n\langle\delta_n, u_n-x_{n+1}\rangle = \|u_n-x_{n+1}\|^2+\frac{\psi^2}{\beta^2}\gamma_n^2\|\delta_n\|^2-\|u_n-x_{n+1}-\frac{\psi}{\beta}\gamma_n\delta_n\|^2. \tag{3.9}$$

Combining the definition of δ_n and (3.2), we observe

$$\begin{aligned}\|\delta_n\| &\le \|u_n-d_n\|+\beta\chi_n\|Qu_n-Qd_n\|\\ &\le (1+\beta\eta)\|u_n-d_n\|.\end{aligned}$$

This together with the definition of γ_n infers that

$$\gamma_n^2\|\delta_n\|^2 = (1-\beta\eta)^2\frac{\|u_n-d_n\|^4}{\|\delta_n\|^2} \ge \frac{(1-\beta\eta)^2}{(1+\beta\eta)^2}\|u_n-d_n\|^2. \tag{3.10}$$

Combining (3.4), (3.7), (3.8), (3.9), and (3.10), we conclude that

$$\|x_{n+1}-x^\dagger\|^2 \le \|u_n-x^\dagger\|^2-\|u_n-x_{n+1}-\frac{\psi}{\beta}\gamma_n\delta_n\|^2-\frac{\psi}{\beta^2}(2\beta-\psi)\frac{(1-\beta\eta)^2}{(1+\beta\eta)^2}\|u_n-d_n\|^2.$$

This completes the proof. □

Theorem 3.1. *Assume Conditions (C1)–(C3) hold. Then the sequence $\{x_n\}$ formed by Algorithm 3.1 converges strongly to $x^\dagger \in \Omega$, where $\|x^\dagger\| = \min\{\|z\| : z \in \Omega\}$.*

Proof. To begin with, our first goal is to show that sequence $\{x_n\}$ is bounded. In view of $\psi \in (0, 2/\eta)$, $\beta \in (\psi/2, 1/\eta)$, and Lemma 3.3, we have

$$\|x_{n+1}-x^\dagger\| \le \|u_n-x^\dagger\|, \quad \forall n \ge 1. \tag{3.11}$$

It follows from the definition of u_n that

$$\begin{aligned}\left\|u_n-x^\dagger\right\| &= \left\|(1-\tau_n)\left(x_n-x^\dagger\right)+(1-\tau_n)\alpha_n(x_n-x_{n-1})-\tau_n x^\dagger\right\|\\ &\le (1-\tau_n)\left\|x_n-x^\dagger\right\|+(1-\tau_n)\alpha_n\|x_n-x_{n-1}\|+\tau_n\|x^\dagger\|\\ &= (1-\tau_n)\left\|x_n-x^\dagger\right\|+\tau_n\left[(1-\tau_n)\frac{\alpha_n}{\tau_n}\|x_n-x_{n-1}\|+\|x^\dagger\|\right].\end{aligned} \tag{3.12}$$

By using the definition of α_n and Condition (C3), we deduce

$$\frac{\alpha_n}{\tau_n}\|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty. \tag{3.13}$$

Therefore

$$\lim_{n\to\infty}\left[(1-\tau_n)\frac{\alpha_n}{\tau_n}\|x_n - x_{n-1}\| + \|x^\dagger\|\right] = \|x^\dagger\|.$$

Thus, there exists a constant $M_1 > 0$ such that

$$(1-\tau_n)\frac{\alpha_n}{\tau_n}\|x_n - x_{n-1}\| + \|x^\dagger\| \le M_1, \quad \forall n \ge 1. \tag{3.14}$$

Combining (3.12) and (3.14), we obtain

$$\|u_n - x^\dagger\| \le (1-\tau_n)\|x_n - x^\dagger\| + \tau_n M_1, \quad \forall n \ge 1. \tag{3.15}$$

From (3.11) and (3.15), we deduce

$$\begin{aligned}\|x_{n+1} - x^\dagger\| &\le (1-\tau_n)\|x_n - x^\dagger\| + \tau_n M_1\\ &\le \max\{\|x_n - x^\dagger\|, M_1\}, \quad \forall n \ge 1\\ &\le \cdots \le \max\{\|x_1 - x^\dagger\|, M_1\}.\end{aligned}$$

This means that $\{x_n\}$ is bounded, so are $\{u_n\}$ and $\{d_n\}$. In view of (3.15), one sees that

$$\begin{aligned}\|u_n - x^\dagger\|^2 &\le \left[(1-\tau_n)\|x_n - x^\dagger\| + \tau_n M_1\right]^2\\ &= (1-\tau_n)^2\|x_n - x^\dagger\|^2 + \tau_n\left[2(1-\tau_n)M_1\|x_n - x^\dagger\| + \tau_n M_1^2\right]\\ &\le \|x_n - x^\dagger\|^2 + \tau_n M_2, \quad \forall n \ge 1,\end{aligned} \tag{3.16}$$

where $M_2 := \sup_{n\in\mathbb{N}}\{2(1-\tau_n)M_1\|x_n - x^\dagger\| + \tau_n M_1^2\} > 0$. From Lemma 3.3 and (3.16), we derive

$$\begin{aligned}&\|u_n - x_{n+1} - \frac{\psi}{\beta}\gamma_n\delta_n\|^2 - \frac{\psi}{\beta^2}(2\beta - \psi)\frac{(1-\beta\eta)^2}{(1+\beta\eta)^2}\|u_n - d_n\|^2\\ &\le \|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 + \tau_n M_2, \quad \forall n \ge 1.\end{aligned} \tag{3.17}$$

By using the definition of u_n and (3.11), we obtain

$$\begin{aligned}\|x_{n+1} - x^\dagger\|^2 &\le \|(1-\tau_n)(x_n - x^\dagger) + (1-\tau_n)\alpha_n(x_n - x_{n-1}) - \tau_n x^\dagger\|^2\\ &\le \|(1-\tau_n)(x_n - x^\dagger) + (1-\tau_n)\alpha_n(x_n - x_{n-1})\|^2 + 2\tau_n\left\langle -x^\dagger, u_n - x^\dagger\right\rangle\\ &\le (1-\tau_n)^2\|x_n - x^\dagger\|^2 + 2(1-\tau_n)\alpha_n\|x_n - x^\dagger\|\|x_n - x_{n-1}\|\\ &\quad + \alpha_n^2\|x_n - x_{n-1}\|^2 + 2\tau_n\left\langle -x^\dagger, u_n - x_{n+1}\right\rangle + 2\tau_n\left\langle -x^\dagger, x_{n+1} - x^\dagger\right\rangle\end{aligned}$$

for all $n \ge 1$. Thus

$$\begin{aligned}\|x_{n+1} - x^\dagger\|^2 \le (1-\tau_n)\|x_n - x^\dagger\|^2 + \tau_n\Big[&2(1-\tau_n)\|x_n - x^\dagger\|\frac{\alpha_n}{\tau_n}\|x_n - x_{n-1}\|\\ &+ \alpha_n\|x_n - x_{n-1}\|\frac{\alpha_n}{\tau_n}\|x_n - x_{n-1}\| + 2\|x^\dagger\|\|u_n - x_{n+1}\|\\ &+ 2\langle x^\dagger, x^\dagger - x_{n+1}\rangle\Big], \quad \forall n \ge 1.\end{aligned} \tag{3.18}$$

Finally, we show that $\{\|x_n - x^\dagger\|\}$ converges to zero. By Lemma 2.1, we assume that $\{\|x_{n_k} - x^\dagger\|^2\}$ is a subsequence of $\{\|x_n - x^\dagger\|^2\}$ such that

$$\liminf_{k\to\infty}\left(\|x_{n_k+1} - x^\dagger\|^2 - \|x_{n_k} - x^\dagger\|^2\right) \geq 0.$$

Note that $2\beta - \psi > 0$ for all $n \geq 1$. Combining Condition (C3) and (3.17), we deduce

$$\begin{aligned}
&\limsup_{k\to\infty}\left(\frac{\psi}{\beta^2}(2\beta-\psi)\frac{(1-\beta\eta)^2}{(1+\beta\eta)^2}\|u_{n_k} - d_{n_k}\|^2 + \|u_{n_k} - x_{n_k+1} - \frac{\psi}{\beta}\gamma_{n_k}\delta_{n_k}\|^2\right)\\
&\leq \limsup_{k\to\infty}\left[\|x_{n_k} - x^\dagger\|^2 - \|x_{n_k+1} - x^\dagger\|^2\right] + \limsup_{k\to\infty}\tau_{n_k}M_2\\
&= -\liminf_{k\to\infty}\left[\|x_{n_k+1} - x^\dagger\|^2 - \|x_{n_k} - x^\dagger\|^2\right]\\
&\leq 0,
\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|d_{n_k} - u_{n_k}\| = 0 \;\text{ and }\; \lim_{k\to\infty}\|u_{n_k} - x_{n_k+1} - \frac{\psi}{\beta}\gamma_{n_k}\delta_{n_k}\| = 0.$$

From the definition of d_n and (3.2), we obtain

$$\|\delta_{n_k}\| \geq (1-\beta\eta)\|u_{n_k} - d_{n_k}\|.$$

This combining with the definition of γ_n infers that

$$\begin{aligned}
\|u_{n_k} - x_{n_k+1}\| &\leq \|u_{n_k} - x_{n_k+1} - \frac{\psi}{\beta}\gamma_{n_k}\delta_{n_k}\| + \frac{\psi}{\beta}\gamma_{n_k}\|\delta_{n_k}\|\\
&= \|u_{n_k} - x_{n_k+1} - \frac{\psi}{\beta}\gamma_{n_k}\delta_{n_k}\| + \frac{\psi}{\beta}(1-\beta\eta)\frac{\|u_{n_k} - d_{n_k}\|^2}{\|\delta_{n_k}\|}\\
&\leq \|u_{n_k} - x_{n_k+1} - \frac{\psi}{\beta}\gamma_{n_k}\delta_{n_k}\| + \frac{\psi}{\beta}\|u_{n_k} - d_{n_k}\|.
\end{aligned}$$

Thus we obtain $\lim_{k\to\infty}\|x_{n_k+1} - u_{n_k}\| = 0$, which together with the boundedness of $\{x_n\}$ yields

$$\lim_{k\to\infty}\|u_{n_k} - x_{n_k+1}\|\|x^\dagger\| = 0. \tag{3.19}$$

It follows from the definition of u_n that

$$\begin{aligned}
\|x_{n_k} - u_{n_k}\| &= \|(1-\tau_{n_k})\alpha_{n_k}(x_{n_k} - x_{n_k-1}) - \tau_{n_k}x_{n_k}\|\\
&\leq \|(1-\tau_{n_k})\alpha_{n_k}(x_{n_k} - x_{n_k-1})\| + \|\tau_{n_k}x_{n_k}\|\\
&= \tau_{n_k}\left[(1-\tau_{n_k})\frac{\alpha_{n_k}}{\tau_{n_k}}\|x_{n_k} - x_{n_k-1}\| + \|x_{n_k}\|\right].
\end{aligned}$$

This combining with (3.14) implies $\lim_{k\to\infty}\|x_{n_k} - u_{n_k}\| = 0$. Consequently, we conclude that

$$\|x_{n_k+1} - x_{n_k}\| \leq \|x_{n_k+1} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0 \text{ as } k \to \infty. \tag{3.20}$$

Since the sequence $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup z$ when $j \to \infty$. Furthermore,

$$\limsup_{k\to\infty}\langle x^\dagger, x^\dagger - x_{n_k}\rangle = \lim_{j\to\infty}\langle x^\dagger, x^\dagger - x_{n_{k_j}}\rangle = \langle x^\dagger, x^\dagger - z\rangle. \tag{3.21}$$

Since $\lim_{k\to\infty}\|x_{n_k}-u_{n_k}\|=0$, we obtain $u_{n_k}\rightharpoonup z$. In the light of Lemma 3.2, this together with $\lim_{k\to\infty}\|u_{n_k}-d_{n_k}\|=0$ yields that $z\in\Omega$. From the definition of $x^\dagger$, (2.2), and (3.21), we deduce

$$\limsup_{k\to\infty}\langle x^\dagger,x^\dagger-x_{n_k}\rangle=\langle x^\dagger,x^\dagger-z\rangle\leq 0. \tag{3.22}$$

By using (3.20) and (3.22), we arrive at

$$\limsup_{k\to\infty}\langle x^\dagger,x^\dagger-x_{n_k+1}\rangle\leq\limsup_{k\to\infty}\langle x^\dagger,x^\dagger-x_{n_k}\rangle\leq 0. \tag{3.23}$$

Combining (3.13), (3.18), (3.19), (3.23), and Lemma 2.1, we conclude that $x_n\to x^\dagger$ as $n\to\infty$. That is the required conclusion. □

Next, we present the other modified projection and contraction algorithm that involves only one projection in each iteration. Indeed, this iterative scheme is shown in Algorithm 3.2 below.

Algorithm 3.2 A novel modified inertial projection and contraction method

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Initialization: Take $\alpha>0$, $\rho>0$, $\ell\in(0,1)$, $\eta\in(0,1)$, $\psi\in(0,2)$, and $\beta\in(0,1/\eta)$. Select sequences $\{\zeta_n\}$ and $\{\tau_n\}$ to satisfy Condition (C3). Let $x_0,x_1\in\mathscr{H}$.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n\geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $u_n=(1-\tau_n)(x_n+\alpha_n(x_n-x_{n-1}))$, where α_n is defined in (3.1).
Step 2. Compute $d_n=Proj_{\mathscr{C}}(u_n-\beta\chi_nQu_n)$, where χ_n is defined in (3.2). If $u_n=d_n$, then stop and d_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $x_{n+1}=u_n-\psi\gamma_n\delta_n$, where γ_n and δ_n are defined in (3.3).
Set $n=n+1$ and go to **Step 1**.

The following lemma plays a crucial role in studying the convergence of Algorithm 3.2.

Lemma 3.4. *Assume Condition (C2) holds. Let sequences $\{u_n\}$, $\{d_n\}$, and $\{x_{n+1}\}$ be created by Algorithm 3.2. Then*

$$\|x_{n+1}-x^\dagger\|^2\leq\|u_n-x^\dagger\|^2-\frac{2-\psi}{\psi}\|u_n-x_{n+1}\|^2,\quad\forall x^\dagger\in\Omega$$

and

$$\|u_n-d_n\|^2\leq\left[\frac{1+\beta\eta}{(1-\beta\eta)\psi}\right]^2\|u_n-x_{n+1}\|^2.$$

Proof. It follows from the definition of x_{n+1} that

$$\|x_{n+1}-x^\dagger\|^2=\|u_n-x^\dagger\|^2-2\psi\gamma_n\langle u_n-x^\dagger,\delta_n\rangle+\psi^2\gamma_n^2\|\delta_n\|^2. \tag{3.24}$$

Combining (3.2) and (3.3), we deduce

$$\begin{aligned}&\langle u_n-x^\dagger,\delta_n\rangle\\&=\langle u_n-d_n,u_n-d_n-\beta\chi_n(Qu_n-Qd_n)\rangle+\langle d_n-x^\dagger,\delta_n\rangle\\&\geq\|u_n-d_n\|^2-\beta\chi_n\|u_n-d_n\|\|Qu_n-Qd_n\|+\langle d_n-x^\dagger,\delta_n\rangle\\&\geq(1-\beta\eta)\|u_n-d_n\|^2+\langle d_n-x^\dagger,u_n-d_n-\beta\chi_n(Qu_n-Qd_n)\rangle.\end{aligned} \tag{3.25}$$

From the definition of d_n and (2.2), we obtain

$$\langle u_n-d_n-\beta\chi_nQu_n,d_n-x^\dagger\rangle\geq 0. \tag{3.26}$$

According to $x^\dagger \in \Omega$ and $d_n \in \mathscr{C}$, we obtain $\langle Qx^\dagger, d_n - x^\dagger\rangle \geq 0$. This together with the pseudomonotonicity of Q infers that

$$\langle Qd_n, d_n - x^\dagger\rangle \geq 0. \tag{3.27}$$

It follows from (3.3) that

$$(1-\beta\eta)\|u_n - d_n\|^2 = \gamma_n\|\delta_n\|^2.$$

This combining with (3.25), (3.26), and (3.27) deduces that

$$\langle u_n - x^\dagger, \delta_n\rangle \geq (1-\beta\eta)\|u_n - d_n\|^2 = \gamma_n\|\delta_n\|^2. \tag{3.28}$$

Note that $x_{n+1} - u_n = \psi\gamma_n\delta_n$. By using (3.24) and (3.28), we conclude that

$$\begin{aligned}\|x_{n+1} - x^\dagger\|^2 &\leq \|u_n - x^\dagger\|^2 - 2\psi\gamma_n^2\|\delta_n\|^2 + \psi^2\gamma_n^2\|\delta_n\|^2 \\ &= \|u_n - x^\dagger\|^2 - \frac{2-\psi}{\psi}\|u_n - x_{n+1}\|^2.\end{aligned}$$

It follows from the definition of x_{n+1} and (3.3) that

$$\|u_n - d_n\|^2 = \frac{1}{\gamma_n(1-\beta\eta)}\|\gamma_n\delta_n\|^2 = \frac{1}{\gamma_n(1-\beta\eta)\psi^2}\|u_n - x_{n+1}\|^2. \tag{3.29}$$

Since $\|\delta_n\| \leq (1+\beta\eta)\|u_n - d_n\|$, we derive

$$\gamma_n = (1-\beta\eta)\frac{\|u_n - d_n\|^2}{\|\delta_n\|^2} \geq \frac{1-\beta\eta}{(1+\beta\eta)^2}. \tag{3.30}$$

Combining (3.29) and (3.30), we arrive at

$$\|u_n - d_n\|^2 \leq \left[\frac{1+\beta\eta}{(1-\beta\eta)\psi}\right]^2 \|u_n - x_{n+1}\|^2.$$

The proof is completed. □

Theorem 3.2. *Assume Conditions (C1)–(C3) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.2 converges strongly to $x^\dagger \in \Omega$, where $\|x^\dagger\| = \min\{\|z\| : z \in \Omega\}$.*

Proof. The proof is similar to that of Theorem 3.1, and so we omit some details of the proof. In view of $\psi \in (0,2)$ and Lemma 3.4, we deduce

$$\|x_{n+1} - x^\dagger\| \leq \|u_n - x^\dagger\|, \quad \forall n \geq 1. \tag{3.31}$$

Utilizing the same arguments as declared in Theorem 3.1, we observe that $\{u_n\}$, $\{d_n\}$, and $\{x_{n+1}\}$ are bounded. It follows from (3.16) and Lemma 3.4 that

$$\frac{2-\psi}{\psi}\|u_n - x_{n+1}\|^2 \leq \|x_n - x^\dagger\|^2 - \|x_{n+1} - x^\dagger\|^2 + \tau_n M_2, \quad \forall n \geq 1. \tag{3.32}$$

Furthermore, we can obtain (3.18) by applying the same facts as stated in Theorem 3.1. Finally, we show that $\{\|x_n - x^\dagger\|\}$ converges to zero. From Condition (C3) and (3.32), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\frac{2-\psi}{\psi}\|u_{n_k} - x_{n_k+1}\|^2 &\leq \limsup_{k\to\infty}\left[\|x_{n_k} - x^\dagger\|^2 - \|x_{n_k+1} - x^\dagger\|^2 + \tau_{n_k}M_2\right] \\ &\leq 0.\end{aligned}$$

This implies that $\lim_{k\to\infty}\|x_{n_k+1} - u_{n_k}\| = 0$, which together with Lemma 3.4 yields $\lim_{k\to\infty}\|d_{n_k} - u_{n_k}\| = 0$. As asserted in Theorem 3.1, we can obtain the same result as (3.19)–(3.23). Therefore, we conclude that $x_n \to x^\dagger$ as $n \to \infty$. This completes the proof. □

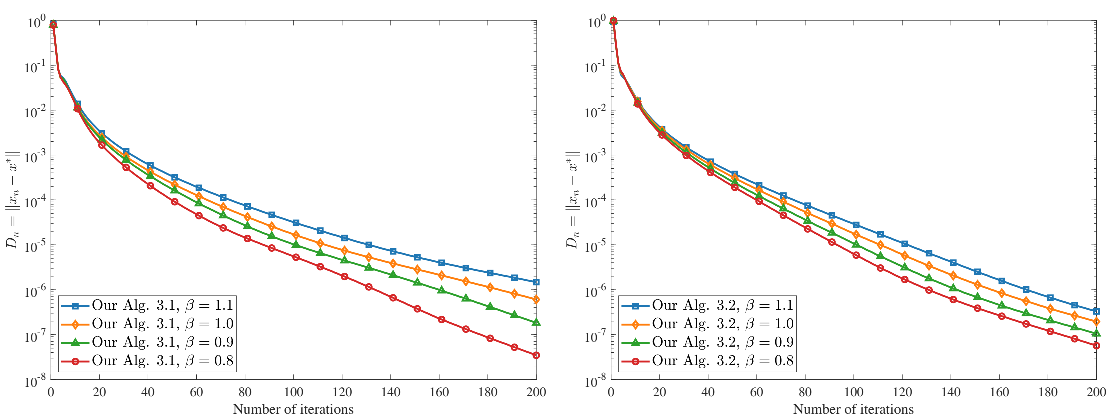
4. NUMERICAL EXPERIMENTS

In this section, we provide some numerical examples to illustrate the advantages and efficiency of the proposed two algorithms compared to some known ones. All the programs are implemented in MATLAB 2018a on a personal computer.

Example 4.1. Let operator $Q:\mathbb{R}^m \to \mathbb{R}^m$ be given by $Q(x) = Gx$, where $G = BB^{\mathsf{T}} + S + E$, matrix $B \in \mathbb{R}^{m\times m}$, matrix $S \in \mathbb{R}^{m\times m}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{m\times m}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). Let the feasible set $\mathscr{C}$ be a box constraint with the form $\mathscr{C} = [-2,5]^m$. It is easy to see that Q is monotone and Lipschitz continuous with constant $L = \|G\|$. In this example, all entries of B, S are generated randomly in $[-2,2]$ and E is generated randomly in $[0,2]$. The solution set of the (VIP) with Q and $\mathscr{C}$ given above is $x^* = \{\mathbf{0}\}$. We compare the proposed methods with Algorithms 3.1 and 3.2 presented by Thong and Gibali [15] (shortly, TG Alg. 3.1 and TG Alg. 3.2) and Algorithms 3.1 and 3.2 introduced by Gibali et al. [17] (shortly, GTT Alg. 3.1 and GTT Alg. 3.2). The parameters of all algorithms are set as follows.

- Choose $\rho = 2$, $\ell = 0.5$, $\eta = 0.6$, $\psi = 1.5$, and $\tau_n = 1/(n+1)$ for all algorithms.
- Take $\alpha = 0.4$, $\zeta_n = 100/(n+1)^2$, and $\beta = 0.8$ for the proposed algorithms.
- Set $\sigma_n = 0.5(1-\tau_n)$ for TG Alg. 3.1 [15] and GTT Alg. 3.1 [17]. Pick $f(x) = 0.1x$ for TG Alg. 3.2 [15] and GTT Alg. 3.2 [17].

The maximum number of iterations 200 is used as a common stopping criterion and the function $D_n = \|x_n - x^*\|$ is applied to measure the nth iteration error of all algorithms. The numerical performance of the proposed algorithms with different parameters β in dimension $m = 20$ is shown in Figure 1. In addition, the numerical results of all algorithms with three different dimensions are reported in Table 1.



(a) The behavior of Our Algorithm 3.1 (b) The behavior of Our Algorithm 3.2

FIGURE 1. The behavior of our algorithms with different β in Example 4.1 ($m = 20$)

Example 4.2. Let a Hilbert space $\mathscr{H} = L^2([0,1])$ be associated with the inner product

$$\langle x,y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t, \quad \forall x,y \in \mathscr{H}$$

TABLE 1. Numerical results of all algorithms for Example 4.1

Algorithms	$m=20$		$m=50$		$m=100$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	7.80E-09	0.0281	2.10E-06	0.033	1.52E-05	0.0657
Our Alg. 3.2	1.12E-07	0.0281	4.44E-06	0.0316	2.38E-05	0.0765
TG Alg. 3.1 [15]	9.31E-04	0.0378	3.16E-03	0.0601	5.99E-03	0.0825
TG Alg. 3.2 [15]	4.06E-04	0.0388	2.27E-03	0.0335	6.10E-03	0.0781
GTT Alg. 3.1 [17]	9.31E-04	0.0278	3.16E-03	0.0321	5.99E-03	0.0744
GTT Alg. 3.2 [17]	4.06E-04	0.0329	2.27E-03	0.0363	6.10E-03	0.0618

and the induced norm

$$\|x\| = \left(\int_0^1 |x(t)|^2 \mathrm{d}t\right)^{1/2}, \quad \forall x \in \mathscr{H}.$$

Let the feasible set be given by $\mathscr{C} = \{x \in \mathscr{H} : \|x\| \leq 1\}$. Define an operator $Q : \mathscr{C} \to \mathscr{H}$ by

$$(Qx)(t) = \int_0^1 (x(t) - W(t,s)P(x(s)))\mathrm{d}s + g(t), \quad t \in [0,1], x \in \mathscr{C},$$

where

$$W(t,s) = \frac{2tse^{t+s}}{e\sqrt{\mathrm{e}^2-1}}, \quad P(x) = \cos x, \quad g(t) = \frac{2te^t}{e\sqrt{\mathrm{e}^2-1}}.$$

It is known that Q is monotone and L-Lipschitz continuous with $L=2$ (see [10]), and $x^*(t) = \{\mathbf{0}\}$ is the solution of the (VIP). We also compare the proposed algorithms with the ones mentioned in Example 4.1. The parameters of all algorithms are set as follows.

- Choose $\rho = 2$, $\ell = 0.5$, $\eta = 0.4$, $\psi = 1.5$, and $\tau_n = 1/(n+1)$ for all the algorithms.
- Adopt $\alpha = 0.2$, $\zeta_n = 1/(n+1)^2$, and $\beta = 0.8$ for the proposed algorithms.
- Set $\sigma_n = 0.9(1-\tau_n)$ for TG Alg. 3.1 [15] and GTT Alg. 3.1 [17]. Pick $f(x) = 0.1x$ for TG Alg. 3.2 [15] and GTT Alg. 3.2 [17].

The function $D_n = \|x_n(t) - x^*(t)\|$ is used to measure nth iteration error of all algorithms and the maximum number of iterations 20 is applied as a common stopping criterion. The numerical results of all algorithms with three different initial values $x_0(t) = x_1(t)$ are displayed in Table 2.

TABLE 2. Numerical results of all algorithms for Example 4.2

Algorithms	$x_1(t) = 2t^2$		$x_1(t) = 2\sin(t)$		$x_1(t) = 2\log(t)$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	4.72E-08	24.5950	2.80E-08	25.0347	4.91E-07	26.2779
Our Alg. 3.2	6.79E-07	23.5687	1.32E-06	23.5672	8.37E-06	25.1187
TG Alg. 3.1 [15]	5.78E-05	22.8222	6.82E-05	22.9238	8.53E-05	24.4893
TG Alg. 3.2 [15]	3.99E-05	22.6094	4.72E-05	23.0316	5.91E-05	24.5711
GTT Alg. 3.1 [17]	5.78E-05	20.9109	6.82E-05	21.7933	3.55E-05	23.9774
GTT Alg. 3.2 [17]	3.99E-05	21.5904	4.72E-05	21.5714	2.32E-05	23.9428

Example 4.3. Consider a Hilbert space

$$\mathscr{H} = l_2 = \{x = (x_1, x_2, \ldots, x_i, \ldots) \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$$

equipped with inner product

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i, \quad \forall x, y \in \mathscr{H}$$

and norm

$$\|x\| = \sqrt{\langle x, x\rangle}, \quad \forall x \in \mathscr{H}.$$

Let the feasible set by defined as $\mathscr{C} = \{x \in \mathscr{H} : |x_i| \leq 1/i\}$. Define an operator $Q : \mathscr{C} \to \mathscr{H}$ by

$$Qx = \left(\|x\| + \frac{1}{\|x\| + \varphi}\right)x, \quad \varphi > 0.$$

It can be checked that Q is pseudomonotone on $\mathscr{H}$, uniformly continuous and sequentially weakly continuous on $\mathscr{C}$ but not Lipschitz continuous on $\mathscr{H}$ (see [27] for more details). In this test, we choose $\varphi = 1.0$ and $\mathscr{H} = \mathbb{R}^m$ for different values of m. We compare the proposed algorithms with Algorithm 3 offered by Thong et al. [22] (shortly, TSI Alg. 3) and Algorithm 3.1 presented by Cai et al. [23] (shortly, CDP Alg. 3.1). The parameters of all algorithms are set as follows.

- Choose $\rho = 2$, $\ell = 0.5$, $\eta = 0.6$, and $\tau_n = 1/(n+1)$ for all algorithms.
- Take $\psi = 1.5$, $\alpha = 0.4$, $\zeta_n = 100/(n+1)^2$, and $\beta = 0.8$ for the proposed algorithms.
- Select $f(x) = 0.1x$ for TSI Alg. 3 [22] and CDP Alg. 3.1 [23].

The maximum number of iterations 200 is used as a common stopping criterion. The numerical results of $E_n = \|x_n - x_{n-1}\|$ for all algorithms with three dimensions are reported in Table 3.

TABLE 3. Numerical results of all algorithms for Example 4.3

Algorithms	$m = 1000$		$m = 10000$		$m = 100000$	
	E_n	CPU (s)	E_n	CPU (s)	E_n	CPU (s)
Our Alg. 3.1	6.32E-86	0.0326	3.25E-86	0.1343	2.62E-86	0.8648
Our Alg. 3.2	6.18E-72	0.0390	3.14E-72	0.1243	1.73E-72	0.7884
CDP Alg. 3.1 [23]	9.60E-28	0.0418	1.59E-27	0.1830	1.17E-27	1.0609
TSI Alg. 3 [22]	2.87E-13	0.0305	1.73E-13	0.1603	2.69E-13	0.8745

Next, we use the proposed algorithms to solve the (VIP) that appears in optimal control problems. We recommend readers to refer to [1, 28] for detailed description of the problem.

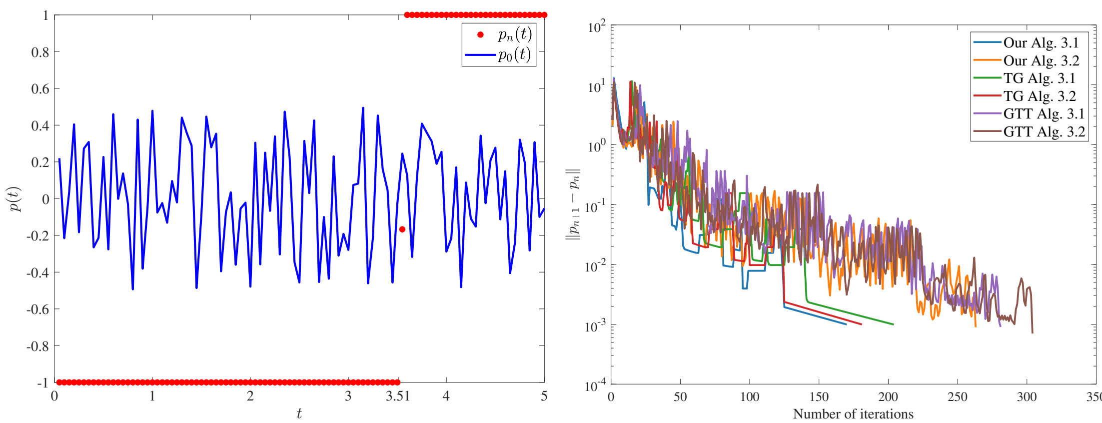
Example 4.4 (Rocket car [28])**.**

$$\begin{aligned}
\text{minimize} \quad & \frac{1}{2}\left((x_1(5))^2 + (x_2(5))^2\right), \\
\text{subject to} \quad & \dot{x}_1(t) = x_2(t), \ \dot{x}_2(t) = p(t), \ \forall t \in [0,5], \\
& x_1(0) = 6, \ x_2(0) = 1, \ p(t) \in [-1,1].
\end{aligned}$$

The exact optimal control of Example 4.4 is $p^*(t) = -1$ if $t \in (0, 3.517]$ and $p^*(t) = 1$ if $t \in (3.517, 5]$. We compare the proposed algorithms with the ones in [15, 17]. The parameters of all algorithms are set as follows.

- Take $\rho = 1$, $\ell = 0.5$, $\eta = 0.4$, $\psi = 1.5$, and $\tau_n = 10^{-4}/(n+1)$ for all algorithms.
- Pick $\beta = 0.8$, $\alpha = 0.01$, and $\zeta_n = 10^{-4}/(n+1)^2$ for the proposed algorithms.
- Set $\sigma_n = 0.9(1-\tau_n)$ for TG Alg. 3.1 [15] and GTT Alg. 3.1 [17]. Choose $f(x) = 0.1x$ for TG Alg. 3.2 [15] and GTT Alg. 3.2 [17].

The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$ and the stopping criterion is $\|p_{n+1} - p_n\| \le 10^{-3}$. The approximate optimal control of the proposed Algorithm 3.1 is plotted in Figure 2(a). In addition, the numerical results of all algorithms are reported in Figure 2(b).



(a) The optimal control of our Algorithm 3.1

(b) Compare all algorithms

FIGURE 2. Numerical results for Example 4.4

Remark 4.1. We have the following observations for Examples 4.1–4.4.

(1) It can be seen from Figure 1 that our two projection and contraction algorithms inserted with a new parameter β are efficient. That is, the proposed iterative methods can obtain a higher accuracy if the appropriate value of β is chosen.
(2) As shown in the numerical results of Examples 4.1–4.4, the two proposed algorithms have a higher accuracy and faster convergence speed than some known schemes in the literature [15, 17, 22, 23] when performing the same stopping criterion, and more importantly, these observations are not related to the choice of the initial values and the size of the dimensions (cf. Table 1, Table 2, Table 3, and Figure 2(b)).
(3) Notice that the operator Q in Example 4.3 is pseudomonotone and uniformly continuous rather than Lipschitz continuous. In this case, the methods introduced in [15, 16, 17] for solving monotone and Lipschitz continuous (or even non-Lipschitz continuous) variational inequalities and the algorithms offered in [18, 19, 20, 21] for solving pseudomonotone and Lipschitz continuous variational inequalities will not be available.

Therefore, the algorithms presented in this paper are efficient and robust.

5. Conclusions

In this paper, we introduced two new iterative algorithms to solve pseudomonotone and non-Lipschitz continuous variational inequality problems in real Hilbert spaces. The proposed methods require computing the projection on the feasible set only once in each iteration. The strong convergence of the iterative sequence generated by the proposed algorithms was proved without the prior knowledge of the Lipschitz constant of the operator. Finally, some numerical examples in finite- and infinite-dimensional spaces and applications in optimal control problems were given to illustrate the advantages and efficiency of our algorithms. The results obtained in this paper improved and extended many known ones in the field.

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