Advances in Operator Theory (2021)6:61
https://doi.org/10.1007/s43036-021-00155-0

Tusi Mathematical Research Group

ORIGINAL PAPER

Inertial extragradient algorithms with non-monotonic step sizes for solving variational inequalities and fixed point problems

Bing Tan[1] · **Jingjing Fan**[1] · **Xiaolong Qin**[2]

Received: 10 March 2021 / Accepted: 28 June 2021

Abstract

In this paper, we introduce four inertial extragradient algorithms with non-monotonic step sizes to find the solution of the convex feasibility problem, which consists of a monotone variational inequality problem and a fixed point problem with a demicontractive mapping. Strong convergence theorems of the suggested algorithms are established under some standard conditions. Finally, we implement some computational tests to show the efficiency and advantages of the proposed algorithms and compare them with some existing ones.

Keywords Variational inequality problem · Fixed point problem · Subgradient extragradient method · Tseng's extragradient method · Inertial method · Strong convergence

Mathematics Subject Classification 47H09 · 47J20 · 49J40 · 65J15 · 90C30

Communicated by Guowei Dai.

✉ Xiaolong Qin
qinxl@yuntech.edu.tw

Bing Tan
bingtan72@gmail.com

Jingjing Fan
fanjingjing0324@163.com

[1] Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China

[2] National Yunlin University of Science and Technology, Douliou, Taiwan

Published online: 19 July 2021

1 Introduction

Let C be a nonempty closed convex set in a real Hilbert space $\mathcal{H}$ whose induced norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. One recalls that the variational inequality problem (shortly, VIP) is described as follows:

$$\text{find } u \in C \text{ such that } \langle Au, x-u\rangle \geq 0, \quad \forall x \in C, \tag{VIP}$$

where $A : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator. Let $\mathrm{VI}(C, A)$ represent the solution set of the problem (VIP). Variational inequality is an essential tool for studying many fields of mathematics and applied sciences (such as physics, regional, social, engineering, and other issues); see, for example, [1, 10, 15, 16]. The theories and methods of variational inequalities have been implemented in numerous areas of science and have proven to be successful and creative. The theory has been shown to provide an easy, common and consistent structure for dealing with possible issues. In the past few decades, researchers have been very interested in developing effective and robust numerical approaches for solving variational inequality problems. In particular, there has been great interest in projection-based methods and their variants. To see various projection-type methods, one refers to [2, 6, 11, 17, 23] and the references therein. It should be mentioned that the extragradient method [11] needs to perform two projection calculations on the feasible set in each iteration, while the subgradient extragradient method [2] and the Tseng's extragradient method [23] only require one projection on the feasible set. It is well known that calculating the projection on a non-empty closed convex set is not easy, especially when it has a complex structure. Thus, these two methods greatly improve computational performance in the actual environment.

On the other hand, the fixed point problem is closely related to variational inequalities. A point $u \in \mathcal{H}$ is called a fixed point of mapping $T : \mathcal{H} \to \mathcal{H}$ if $Tu = u$. We use $\mathrm{Fix}(T)$ to denote the fixed point set of T. Our main objective in this paper is to find general solutions to variational inequality problems and fixed point problems. The reason for exploring these problems is that they can be applied to mathematical models, and their constraints can be represented as fixed-point problems and/or variational inequality problems. In recent years, researchers have investigated and proposed many efficient iterative approaches to find common solutions for variational inequality problems and fixed point problems, see, for instance, [3, 7, 18, 29] and the references therein. Recently, Kraikaew and Saejung [12] proposed an algorithm for finding a common solution to monotone variational inequalities and fixed point problems. This algorithm is based on the Halpern method and the subgradient extragradient method, and is now called the Halpern subgradient extragradient method (HSEGM). Indeed, the algorithm is of the form:

$$\begin{cases} y^k = P_C(x^k - \gamma Ax^k), \\ z^k = \theta_k x^0 + (1-\theta_k)P_{H_k}(x^k - \gamma Ay^k), \\ H_k = \left\{x \in \mathcal{H} \mid \langle x^k - \gamma Ax^k - y^k, x - y^k\rangle \le 0\right\}, \\ x^{k+1} = \eta_k x^k + (1-\eta_k)Tz^k, \end{cases} \tag{HSEGM}$$

where P_C stands for the metric projection of $\mathcal{H}$ onto C ($P_C(x) := \operatorname{argmin}\{\|x-y\|, y \in C\}$), mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous, the step size γ is a fixed number and belongs to (0, $1/L$) , and mapping $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive (see below for the definition). They proved that the iterative sequence $\{x^k\}$ defined in (HSEGM) converges to $P_{\mathrm{Fix}(T)\cap \mathrm{VI}(C,A)}(x^0)$ in norm under some suitable conditions. However, Algorithm (HSEGM) needs to know the prior information of the Lipschitz constant of the mapping, which may limit the use of some related algorithms. To overcome such difficulty, a large number of algorithms have been proposed to update the step size through certain adaptive criteria, see, for example, [4, 19, 24]. Recently, Tong and Tian [24] proposed a new self-adaptive iterative algorithm to solve variational inequality problems and fixed point problems in a Hilbert space. Their algorithm is motivated by the Tseng's extragradient method, the hybrid steepest descent method and the Mann-type method. The adaptive criterion adopted can guarantee that the algorithm works without knowing the Lipschitz constant of the mapping. Their algorithm is described as follows:

$$\begin{cases} y^k = P_C(x^k - \gamma_k Ax^k), \\ z^k = y^k - \gamma_k(Ay^k - Ax^k), \\ t^k = (1-\eta_k)z^k + \eta_k Tz^k, \\ x^{k+1} = (I - \lambda\theta_k F)t^k, \end{cases} \tag{STEGM}$$

where mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous, mapping $T : \mathcal{H} \to \mathcal{H}$ is quasi-nonexpansive with a demiclosedness property and mapping $F : \mathcal{H} \to \mathcal{H}$ is strongly monotone and Lipschitz continuous. The step size γ_k will be automatically updated in each iteration by selecting the maximum $\gamma \in \left\{\rho, \rho l, \rho l^2, \ldots\right\}$ that satisfies $\gamma\|Ax^k - Ay^k\| \le \phi\|x^k - y^k\|$ (this rule is called the Armijo-like line search criterion). Under some suitable conditions, the iterative sequence generated by (STEGM) converges to $z = P_{\mathrm{Fix}(T)\cap \mathrm{VI}(C,A)}(I - \gamma F)z$ in norm.

In this paper, we focus on the situation that T is a demicontractive mapping, which covers quasi-nonexpansive mappings. In 2018, Thong and Hieu [25] proposed two Mann-type subgradient extragradient algorithms to find common elements of variational inequalities and fixed point problems involving a demicontractive mapping. More precisely, their iterative algorithms are as follows:

$$\begin{cases} y^k = P_C(x^k - \gamma A x^k), \\ z^k = P_{H_k}(x^k - \gamma A y^k), \\ H_k = \{x \in \mathcal{H} \mid \langle x^k - \gamma A x^k - y^k, x - y^k \rangle \le 0\}, \\ x^{k+1} = (1 - \theta_k - \eta_k) z^k + \eta_k T z^k, \end{cases} \tag{MSEGM}$$

and

$$\begin{cases} y^k = P_C(x^k - \gamma A x^k), \\ z^k = P_{H_k}(x^k - \gamma A y^k), \\ H_k = \{x \in \mathcal{H} \mid \langle x^k - \gamma A x^k - y^k, x - y^k \rangle \le 0\}, \\ x^{k+1} = (1 - \eta_k)(\theta_k z^k) + \eta_k T z^k, \end{cases} \tag{MMSEGM}$$

where mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous, step size $\gamma \in (0, 1/L)$ and mapping $T : \mathcal{H} \to \mathcal{H}$ is λ-demicontractive with $0 \le \lambda < 1$. They obtained strong convergence theorems of the suggested algorithms in real Hilbert spaces under some suitable and mild assumptions.

Note that algorithms (MSEGM) and (MMSEGM) require to know the prior information of the Lipschitz constant of the cost mapping. In addition, we point out that the method of updating the step size through the Armijo-like criterion may be computationally expensive because it needs to calculate the value of operator A and the projection P_C many times in each iteration. To overcome these shortcomings, an effective method is to automatically update the step size through some simple calculations in each iteration. Recently, Thong and Hieu [26] introduced two extragradient viscosity-type iterative algorithms with a new simple step size to solve variational inequalities and fixed point problems. Their algorithms are of the following forms:

$$\begin{cases} y^k = P_C(x^k - \gamma_k A x^k), \\ z^k = P_{H_k}(x^k - \gamma_k A y^k), \\ H_k = \{x \in \mathcal{H} \mid \langle x^k - \gamma_k A x^k - y^k, x - y^k \rangle \le 0\}, \\ x^{k+1} = \theta_k f(x^k) + (1 - \theta_k)\left[(1 - \eta_k) z^k + \eta_k T z^k\right], \end{cases} \tag{VSEGM}$$

and

$$\begin{cases} y^k = P_C(x^k - \gamma_k A x^k), \\ z^k = y^k - \gamma_k (A y^k - A x^k), \\ x^{k+1} = \theta_k f(x^k) + (1 - \theta_k)\left[(1 - \eta_k) z^k + \eta_k T z^k\right], \end{cases} \tag{VTEGM}$$

where mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous, mapping $T : \mathcal{H} \to \mathcal{H}$ is λ-demicontractive and mapping f is contractive. Algorithms (VSEGM) and (VTEGM) update the step size $\{\gamma_k\}$ by the following rule:

$$\gamma_{k+1} = \begin{cases} \min\left\{\dfrac{\phi\|x^k - y^k\|}{\|Ax^k - Ay^k\|}, \gamma_k\right\}, & \text{if } Ax^k - Ay^k \neq 0; \\ \gamma_k, & \text{otherwise} . \end{cases}$$

Note that the step size γ_k updated by the above method is non-increasing, i.e., $\gamma_{k+1} \leq \gamma_k \, (k \geq 1)$. This means that the method may depend on the choice of the initial step size. It should be highlighted that algorithms (STEGM), (VSEGM) and (VTEGM) only need to compute the projection on the feasible set C once in each iteration, and they can work without the prior information of the Lipschitz constant of the cost mapping. These algorithms have achieved strong convergence theorems in real Hilbert spaces under some suitable conditions.

In recent years, the development of fast iterative algorithms has aroused great interest from scientific researchers. The inertial algorithm is a two-stage iterative procedure. Its main feature is to use the previous two iterations to represent the next iteration. Many authors have used inertial methods to build a large number of iterative algorithms that can improve the convergence speed; see, for instance, [5, 8, 9, 20–22, 27, 30] and the references therein. These inertial-type algorithms have a better numerical performance than algorithms without inertial terms.

Motivated and stimulated by results as mentioned above, in this paper, we suggest four new inertial Mann-type extragradient algorithms by inserting the inertial terms into the Tseng's extragradient algorithm and the subgradient extragradient algorithm. They are used to find a common element of the solution set of the monotone variational inequality problem and the fixed point set of a demicontractive mapping. We use a new non-monotonic step size in each iteration, which allows the algorithms to work without knowing the Lipschitz constant of the mapping in advance. We obtain strong convergence of these algorithms under some standard and mild hypotheses. Finally, we give several numerical examples to support the theoretical results. Numerical results show that the new algorithms converge faster than the existing ones [12, 24–26].

The remaining part of the paper proceeds as follows. In the next Section, we recall some preliminary results. In Sect. 3, we analyze the convergence of the proposed algorithms. In Sect. 4, some computational tests are provided to illustrate the numerical behavior of the proposed algorithms and compare them with some existing ones. Finally, a brief summary is given in Sect. 5, the last section.

2 Preliminaries

Throughout this paper, we always assume that $\mathcal{H}$ represents a Hilbert space and C denotes the nonempty convex and closed subset of $\mathcal{H}$. The weak convergence and strong convergence of $\{x^k\}_{k=1}^{\infty}$ to x are represented by $x^k \rightharpoonup x$ and $x^k \to x$, respectively. For each $x, y \in \mathcal{H}$ and $\theta \in \mathbb{R}$, we have the following basic inequalities:

- $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
- $\|\theta x + (1 - \theta)y\|^2 = \theta\|x\|^2 + (1 - \theta)\|y\|^2 - \theta(1 - \theta)\|x - y\|^2$.

It is known that metric projection P_C has the following basic properties:

- $\langle x - P_C(x), y - P_C(x)\rangle \leq 0$, $\forall y \in C$;
- $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle$, $\forall y \in \mathcal{H}$.

Definition 2.1 Suppose that a nonlinear operator $T : \mathcal{H} \to \mathcal{H}$ satisfies $\mathrm{Fix}(T) \neq \emptyset$. If for any $\{x^k\} \subset \mathcal{H}$, $x^k \rightharpoonup x$ and $(I - T)x^k \to 0$ implies that $x \in \mathrm{Fix}(T)$. Then $I - T$ is said to be demiclosed at zero.

Definition 2.2 For any $x, y \in \mathcal{H}, z \in \mathrm{Fix}(A)$, an operator $A : \mathcal{H} \to \mathcal{H}$ is said to be

- *L-Lipschitz continuous* with $L > 0$ if
$$\|Ax - Ay\| \leq L\|x - y\|.$$
- *monotone* if
$$\langle Ax - Ay, x - y\rangle \geq 0.$$
- *quasi-nonexpansive* if
$$\|Ax - z\| \leq \|x - z\|.$$
- λ-*strictly pseudocontractive* with $0 \leq \lambda < 1$ if
$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + \lambda\|(I - A)x - (I - A)y\|^2.$$
- η-*demicontractive* with $0 \leq \eta < 1$ if
$$\|Ax - z\|^2 \leq \|x - z\|^2 + \eta\|(I - A)x\|^2, \tag{2.1}$$
or equivalently
$$\langle Ax - x, x - z\rangle \leq \frac{\eta - 1}{2}\|x - Ax\|^2, \tag{2.2}$$
or equivalently
$$\langle Ax - z, x - z\rangle \leq \|x - z\|^2 + \frac{\eta - 1}{2}\|x - Ax\|^2. \tag{2.3}$$

Remark 2.3 According to the above definitions, we can easily see the following facts:

- Every strictly pseudocontractive mapping with a nonempty fixed point set is demicontractive.
- The type of demicontractive mappings includes the type of quasi-nonexpansive mappings.

The following lemmas are crucial in the proof of convergence of our main results.

Lemma 2.4 [12] *Suppose that* $A : \mathcal{H} \to \mathcal{H}$ *is a monotone and L-Lipschitz continuous mapping. Let* $T = P_C(I - \phi A)$, *where* $\phi > 0$. *If* $\{x^k\} \subset \mathcal{H}$ *satisfying* $x^k \rightharpoonup u$ *and* $x^k - Tx^k \to 0$, *then* $u \in \mathrm{VI}(C, A) = \mathrm{Fix}(T)$.

Lemma 2.5 [14] *Suppose that* $\{b^k\}$ *is a nonnegative sequence. If there exists a subsequence* $\{b^{k_j}\}$ *of* $\{b^k\}$ *satisfies* $b^{k_j} < b^{k_j+1}, \forall j \in \mathbb{N}$. *Then there exists a nondecreasing sequence* $\{m_k\}$ *of* $\mathbb{N}$ *satisfies* $\lim_{k\to\infty} m_k = \infty$. *Moreover, for all (sufficiently large)* $k \in \mathbb{N}$, *the following inequalities are satisfied:* $b^{m_k} \le b^{m_k+1}$ *and* $b^k \le b^{m_k+1}$. *Actually,* m_k *is the largest number* n *in the set* $\{1, 2, \ldots, k\}$ *satisfies* $b^n < b^{n+1}$.

Lemma 2.6 [28] *Suppose that* $\{a^k\}$ *is a nonnegative sequence satisfying* $a^{k+1} \le \theta_k b^k + (1-\theta_k)a^k, \forall k > 0$, *where* $\{\theta_k\} \subset (0, 1)$ *and* $\{b^k\}$ *is a sequence such that* $\sum_{k=0}^{\infty} \theta_k = \infty$ *and* $\limsup_{k\to\infty} b^k \le 0$. *Then* $\lim_{k\to\infty} a^k = 0$.

Lemma 2.7 [25] *Suppose that* $T : \mathcal{H} \to \mathcal{H}$ *is* η*-demicontractive with* $\mathrm{Fix}(T) \ne \emptyset$. *Set* $T_\lambda = \lambda T + (1-\lambda)I$, *where* I *stands for identity mapping and* $\lambda \in (0, 1-\eta)$. *Then*

(i) $\mathrm{Fix}(T) = \mathrm{Fix}(T_\lambda)$;

(ii) $\|T_\lambda x - u\|^2 \le \|x - u\|^2 - \frac{1}{\lambda}(1 - \eta - \lambda)\|(I - T_\lambda)x\|^2, \quad \forall u \in \mathrm{Fix}(T), x \in \mathcal{H}$;

(iii) $\mathrm{Fix}(T)$ is a convex and closed set.

3 Main results

In this section, we present four inertial extragradient approaches to solve variational inequality problems and fixed point problems, and analyze their convergence. These algorithms are inspired and driven by the subgradient extragradient method, the Tseng's extragradient method and the Mann-type method. In particular, we have added an inertial term and a new non-monotonic step size, which makes these algorithms have a faster convergence speed and do not need to know the prior information of Lipschitz constant in advance. First, we assume that our proposed Algorithm 1 and Algorithm 2 satisfy the subsequent four assumptions.

(C1) The mapping $A : \mathcal{H} \to \mathcal{H}$ is monotone and L-Lipschitz continuous on $\mathcal{H}$.

(C2) The mapping $T : \mathcal{H} \to \mathcal{H}$ is λ-demicontractive such that $(I - T)$ is demiclosed at zero.

(C3) The solution set $\mathrm{Fix}(T) \cap \mathrm{VI}(C, A) \ne \emptyset$.

(C4) Let $\{\zeta_k\}$ and $\{\xi_k\}$ be two nonnegative sequences such that $\lim_{k\to\infty} \frac{\zeta_k}{\theta_k} = 0$ and $\sum_{k=1}^{\infty} \xi_k < +\infty$, where $\{\theta_k\}$ is a sequence of (0, 1) , and satisfies

$\sum_{k=1}^{\infty} \theta_k = \infty$ and $\lim_{k\to\infty} \theta_k = 0$. Let the positive sequence $\{\eta_k\}$ satisfy $\eta_k \in (a, b) \subset (0, (1-\lambda)(1-\theta_k))$ for some $a > 0$, $b > 0$.

3.1 The inertial Mann-type subgradient extragradient algorithm

Now, we present an inertial Mann-type subgradient extragradient algorithm to solve variational inequality problems and fixed point problems. The details of the algorithm are described as follows:

Algorithm 1 The inertial Mann-type subgradient extragradient algorithm

---

Initialization: Take $\delta > 0$, $\gamma_1 > 0$, $\phi \in (0, 1)$. Let $x^0, x^1 \in \mathcal{H}$ be two arbitrary initial points.

Iterative Steps: Calculate the next iteration point x^{k+1} as follows:

Step 1. Take two previously known iteration points x^{k-1} and x^k $(k \geq 1)$. Calculate $s^k = x^k + \delta_k(x^k - x^{k-1})$, where

$$\delta_k = \begin{cases} \min\left\{\dfrac{\zeta_k}{\|x^k - x^{k-1}\|}, \delta\right\}, & \text{if } x^k \neq x^{k-1}; \\ \delta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Calculate $y^k = P_C(s^k - \gamma_k A s^k)$.

Step 3. Calculate $z^k = P_{H_k}(s^k - \gamma_k A y^k)$, where the half-space H_k is defined by

$$H_k := \left\{x \in \mathcal{H} \mid \langle s^k - \gamma_k A s^k - y^k, x - y^k\rangle \leq 0\right\}.$$

Step 4. Calculate $x^{k+1} = (1 - \theta_k - \eta_k)z^k + \eta_k T z^k$, and update the step size γ_{k+1} by

$$\gamma_{k+1} = \begin{cases} \min\left\{\dfrac{\phi\|s^k - y^k\|}{\|As^k - Ay^k\|}, \gamma_k + \xi_k\right\}, & \text{if } As^k - Ay^k \neq 0; \\ \gamma_k + \xi_k, & \text{otherwise.} \end{cases} \tag{3.2}$$

Remark 3.1 It follows from (3.1) that

$$\lim_{k\to\infty} \frac{\delta_k}{\theta_k} \|x^k - x^{k-1}\| = 0.$$

Indeed, we get that $\delta_k \|x^k - x^{k-1}\| \leq \zeta_k$ for all $k \geq 0$, which, together with $\lim_{k\to\infty} \frac{\zeta_k}{\theta_k} = 0$ implies that

$$\lim_{k\to\infty} \frac{\delta_k}{\theta_k} \|x^k - x^{k-1}\| \leq \lim_{k\to\infty} \frac{\zeta_k}{\theta_k} = 0.$$

The following two lemmas are very important for the convergence analysis of the algorithms.

Lemma 3.2 *Suppose that Condition* $(C1)$ *holds. Then the sequence* $\{\gamma_k\}$ *generated by* (3.2) *is well defined and* $\lim_{k\to\infty}\gamma_k=\gamma$ *and* $\gamma\in\left[\min\{\frac{\phi}{L},\gamma_1\},\gamma_1+\Xi\right]$, *where* $\Xi=\sum_{k=1}^{\infty}\xi_k$.

Proof Since mapping A is L-Lipschitz continuous, one has

$$\frac{\phi\|s^k-y^k\|}{\|As^k-Ay^k\|}\geq\frac{\phi\|s^k-y^k\|}{L\|s^k-y^k\|}=\frac{\phi}{L},\ \text{if } As^k\neq Ay^k.$$

Thus, $\gamma_k\geq\min\{\frac{\phi}{L},\gamma_1\}$. It follows from the definition of γ_{k+1} that $\gamma_{k+1}\leq\gamma_1+\Xi$. Consequently, the sequence $\{\gamma_k\}$ defined in (3.2) is bounded and $\gamma_k\in\left[\min\{\frac{\phi}{L},\gamma_1\},\gamma_1+\Xi\right]$. For simplicity, we define $\left(\gamma_{k+1}-\gamma_k\right)^+=\max\{0,\gamma_{k+1}-\gamma_k\}$ and $\left(\gamma_{k+1}-\gamma_k\right)^-=\max\{0,-\left(\gamma_{k+1}-\gamma_k\right)\}$. By the definition of $\{\gamma_k\}$, one obtains $\sum_{k=1}^{\infty}\left(\gamma_{k+1}-\gamma_k\right)^+\leq\sum_{k=1}^{\infty}\xi_k<+\infty$, which implies that the series $\sum_{k=1}^{\infty}\left(\gamma_{k+1}-\gamma_k\right)^+$ is convergent. Next we show the convergence of the series $\sum_{k=1}^{\infty}\left(\gamma_{k+1}-\gamma_k\right)^-$. Suppose that $\sum_{k=1}^{\infty}\left(\gamma_{k+1}-\gamma_k\right)^-=+\infty$. Note that $\gamma_{k+1}-\gamma_k=\left(\gamma_{k+1}-\gamma_k\right)^+-\left(\gamma_{k+1}-\gamma_k\right)^-$. Therefore,

$$\gamma_{m+1}-\gamma_1=\sum_{k=1}^{m}\left(\gamma_{k+1}-\gamma_k\right)=\sum_{k=1}^{m}\left(\gamma_{k+1}-\gamma_k\right)^+-\sum_{k=1}^{m}\left(\gamma_{k+1}-\gamma_k\right)^-.$$

Taking $m\to+\infty$ in the above equation, we get $\lim_{m\to+\infty}\gamma_m\to-\infty$. That is a contradiction. Hence, we deduce that $\lim_{k\to\infty}\gamma_k=\gamma$ and $\gamma\in\left[\min\{\frac{\phi}{L},\gamma_1\},\gamma_1+\Xi\right]$. □

Remark 3.3 The idea of the step size γ_k defined in (3.2) is derived from [13]. It is worth noting that the step size γ_k generated in Algorithm 1 is allowed to increase when the iteration increases. Therefore, the use of this type of step size reduces the dependence on the initial step size γ_1. On the other hand, because of $\sum_{k=1}^{\infty}\xi_k<+\infty$, which implies that $\lim_{k\to\infty}\xi_k=0$. Consequently, the step size γ_k may not increase when k is large enough. If $\xi_k=0$, then the step size γ_k in Algorithm 1 is similar to the approaches in [26].

Lemma 3.4 *Suppose that Conditions* $(C1)$ *and* $(C3)$ *hold. Let the sequence* $\{z^k\}$ *be generated by Algorithm* 1. *Then, for all* $u\in\mathrm{VI}(C,A)$,

$$\|z^k-u\|^2\leq\|s^k-u\|^2-\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|y^k-s^k\|^2-\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|z^k-y^k\|^2.$$

Proof By the definition of γ_k, one has

$$\|As^k-Ay^k\|\leq\frac{\phi}{\gamma_{k+1}}\|s^k-y^k\|,\quad\forall k\geq0.$$

Using $u\in\mathrm{VI}(C,A)\subset C\subset H_k$, we have

$$
\begin{aligned}
2\|z^k - u\|^2 &= 2\|P_{H_k}(s^k - \gamma_k A y^k) - P_{H_k} u\|^2 \\
&\le 2\langle z^k - u, s^k - \gamma_k A y^k - u\rangle \\
&= \|z^k - u\|^2 + \|s^k - \gamma_k A y^k - u\|^2 - \|z^k - s^k + \gamma_k A y^k\|^2 \\
&= \|z^k - u\|^2 + \|s^k - u\|^2 + \gamma_k^2 \|A y^k\|^2 - 2\langle s^k - u, \gamma_k A y^k\rangle \\
&\quad - \|z^k - s^k\|^2 - \gamma_k^2 \|A y^k\|^2 - 2\langle z^k - s^k, \gamma_k A y^k\rangle \\
&= \|z^k - u\|^2 + \|s^k - u\|^2 - \|z^k - s^k\|^2 - 2\langle z^k - u, \gamma_k A y^k\rangle,
\end{aligned}
$$

which implies that

$$
\|z^k - u\|^2 \le \|s^k - u\|^2 - \|z^k - s^k\|^2 - 2\langle z^k - u, \gamma_k A y^k\rangle. \tag{3.3}
$$

We have $\langle Au, y^k - u\rangle \ge 0$ since $u \in \mathrm{VI}(C, A)$. In addition, since A is monotone, one obtains $2\gamma_k \langle A y^k - Au, y^k - u\rangle \ge 0$. Thus, adding this item to the right side of (3.3), we get

$$
\begin{aligned}
\|z^k - u\|^2 &\le \|s^k - u\|^2 - \|z^k - s^k\|^2 - 2\langle z^k - u, \gamma_k A y^k\rangle \\
&\quad + 2\gamma_k \langle A y^k - Au, y^k - u\rangle \\
&= \|s^k - u\|^2 - \|z^k - s^k\|^2 + 2\langle y^k - z^k, \gamma_k A y^k\rangle \\
&\quad - 2\gamma_k \langle Au, y^k - u\rangle \\
&\le \|s^k - u\|^2 - \|z^k - s^k\|^2 + 2\gamma_k \langle y^k - z^k, A y^k - A s^k\rangle \\
&\quad + 2\gamma_k \langle A s^k, y^k - z^k\rangle.
\end{aligned} \tag{3.4}
$$

Note that

$$
\begin{aligned}
&2\gamma_k \langle y^k - z^k, A y^k - A s^k\rangle \\
&\le 2\gamma_k \|A y^k - A s^k\| \|y^k - z^k\| \le 2\phi \frac{\gamma_k}{\gamma_{k+1}} \|s^k - y^k\| \|y^k - z^k\| \\
&\le \phi \frac{\gamma_k}{\gamma_{k+1}} \|s^k - y^k\|^2 + \phi \frac{\gamma_k}{\gamma_{k+1}} \|y^k - z^k\|^2.
\end{aligned} \tag{3.5}
$$

Next, we estimate $2\gamma_k \langle A s^k, y^k - z^k\rangle$. It follows from $z^k = P_{H_k}\left(s^k - \gamma_k A y^k\right)$ that $z^k \in H_k$. Moreover,

$$
\langle s^k - \gamma_k A s^k - y^k, z^k - y^k\rangle \le 0,
$$

which implies that

$$
\begin{aligned}
2\gamma_k\langle As^k, y^k - z^k\rangle &\le 2\langle y^k - s^k, z^k - y^k\rangle \\
&= \|z^k - s^k\|^2 - \|y^k - s^k\|^2 - \|z^k - y^k\|^2.
\end{aligned} \tag{3.6}
$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$
\|z^k - u\|^2 \le \|s^k - u\|^2 - \left(1 - \phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|y^k - s^k\|^2 - \left(1 - \phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|z^k - y^k\|^2.
$$

This completes the proof of the lemma. □

Theorem 3.5 *Suppose that Conditions* $(C1)$–$(C4)$ *hold. Then the iterative sequence* $\{x^k\}$ *generated by Algorithm* 1 *converges to* $u \in \operatorname{Fix}(T) \cap \mathrm{VI}(C, A)$ *in norm, where* $\|u\| = \min\{\|p\| : p \in \operatorname{Fix}(T) \cap \mathrm{VI}(C, A)\}$.

Proof It follows from Lemma 2.7 that $\operatorname{Fix}(T)$ is a convex and closed set. Note that $\mathrm{VI}(C, A)$ is also a closed and convex set. According to the definition of u, we have $u = P_{\mathrm{VI}(C,A)\cap\mathrm{Fix}(T)}(0)$. By Lemma 3.2, we get $\lim_{k\to\infty}(1 - \phi\frac{\gamma_k}{\gamma_{k+1}}) = 1 - \phi > 0$, which means that there exists $k_0 \in \mathbb{N}$ such that $(1 - \phi\frac{\gamma_k}{\gamma_{k+1}}) > 0$, $\forall k \ge k_0$. On account of Lemmas 3.2 and 3.4, we deduce that

$$
\|z^k - u\| \le \|s^k - u\|, \quad \forall k \ge k_0. \tag{3.7}
$$

We next divide the proof into four parts.

Claim 1. The sequence $\{x^k\}$ is bounded. According to the definition of x^{k+1}, one has

$$
\begin{aligned}
\|x^{k+1} - u\| &= \|(1 - \theta_k - \eta_k)(z^k - u) + \eta_k(Tz^k - u) - \theta_k u\| \\
&\le \|(1 - \theta_k - \eta_k)(z^k - u) + \eta_k(Tz^k - u)\| + \theta_k\|u\|.
\end{aligned} \tag{3.8}
$$

Combining (2.1), (2.3) and (3.7), we have

$$
\begin{aligned}
&\|(1 - \theta_k - \eta_k)(z^k - u) + \eta_k(Tz^k - u)\|^2 \\
&= (1 - \theta_k - \eta_k)^2\|z^k - u\|^2 + \eta_k^2\|Tz^k - u\|^2 \\
&\quad + 2(1 - \theta_k - \eta_k)\eta_k\langle Tz^k - u, z^k - u\rangle \\
&\le (1 - \theta_k - \eta_k)^2\|z^k - u\|^2 + \eta_k^2\left[\|z^k - u\|^2 + \lambda\|z^k - Tz^k\|^2\right] \\
&\quad + 2(1 - \theta_k - \eta_k)\eta_k\left[\|z^k - u\|^2 - \frac{1 - \lambda}{2}\|z^k - Tz^k\|^2\right] \\
&= (1 - \theta_k)^2\|z^k - u\|^2 + \eta_k(\eta_k - (1 - \lambda)(1 - \theta_k))\|z^k - Tz^k\|^2 \\
&\le (1 - \theta_k)^2\|s^k - u\|^2,
\end{aligned}
$$

which implies that

$$\|(1-\theta_k-\eta_k)(z^k-u)+\eta_k(Tz^k-u)\| \le (1-\theta_k)\|s^k-u\|. \tag{3.9}$$

From the definition of s^k, we can write

$$\|s^k-u\| \le \|x^k-u\| + \theta_k \cdot \frac{\delta_k}{\theta_k}\|x^k-x^{k-1}\|. \tag{3.10}$$

By Remark 3.1, we get that $\frac{\delta_k}{\theta_k}\|x^k-x^{k-1}\| \to 0$ as $k\to\infty$. Thus, there exists a constant $Q_1>0$ such that

$$\frac{\delta_k}{\theta_k}\|x^k-x^{k-1}\| \le Q_1, \quad \forall k\ge 1. \tag{3.11}$$

From (3.7), (3.10) and (3.11), we find that

$$\|z^k-u\| \le \|s^k-u\| \le \|x^k-u\| + \theta_k Q_1, \quad \forall k\ge k_0. \tag{3.12}$$

Combining (3.8), (3.9) and (3.12), we obtain

$$\begin{aligned}\|x^{k+1}-u\| &\le (1-\theta_k)\|s^k-u\| + \theta_k\|u\| \\ &\le (1-\theta_k)\|x^k-u\| + \theta_k(\|u\|+Q_1) \\ &\le \max\left\{\|x^k-u\|, \|u\|+Q_1\right\} \\ &\le \cdots \le \max\{\|x^{k_0}-u\|, \|u\|+Q_1\}.\end{aligned}$$

Thus, the sequence $\{x^k\}$ is bounded. So the sequences $\{s^k\}$ and $\{z^k\}$ are also bounded.

Claim 2.

$$\begin{aligned}&\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|y^k-s^k\|^2 + \left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|z^k-y^k\|^2 \\ &\quad + \eta_k[(1-\lambda)-\eta_k]\|z^k-Tz^k\|^2 \\ &\le \|x^k-u\|^2 - \|x^{k+1}-u\|^2 + \theta_k Q_4.\end{aligned}$$

Indeed, it follows from (3.12) that

$$\begin{aligned}\|s^k-u\|^2 &\le (\|x^k-u\| + \theta_k Q_1)^2 \\ &= \|x^k-u\|^2 + \theta_k(2Q_1\|x^k-u\| + \theta_k Q_1^2) \\ &\le \|x^k-u\|^2 + \theta_k Q_2\end{aligned} \tag{3.13}$$

for some $Q_2>0$. Using (2.2), (3.12), (3.13) and Lemma 3.4, we obtain

$$
\begin{aligned}
\|x^{k+1}-u\|^2 &= \|(z^k-u)+\eta_k(Tz^k-z^k)-\theta_k z^k\|^2 \\
&\le \|(z^k-u)+\eta_k(Tz^k-z^k)\|^2 - 2\theta_k\langle z^k, x^{k+1}-u\rangle \\
&= \|z^k-u\|^2+\eta_k^2\|Tz^k-z^k\|^2+2\eta_k\langle Tz^k-z^k, z^k-u\rangle \\
&\quad + 2\theta_k\langle z^k, u-x^{k+1}\rangle \\
&\le \|z^k-u\|^2+\eta_k^2\|Tz^k-z^k\|^2+\eta_k(\lambda-1)\|z^k-Tz^k\|^2+\theta_k Q_3 \\
&\le \|x^k-u\|^2+\theta_k Q_4-\eta_k[(1-\lambda)-\eta_k]\|z^k-Tz^k\|^2 \\
&\quad -\Big(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\Big)\|y^k-s^k\|^2-\Big(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\Big)\|z^k-y^k\|^2,
\end{aligned}
$$

where $Q_4 = Q_2 + Q_3$. Thus, we can obtain the desired result through a direct calculation.

Claim 3.

$$
\begin{aligned}
\|x^{k+1}-u\|^2 \le (1-\theta_k)\|x^k-u\|^2+\theta_k\Big[2\eta_k\|z^k-Tz^k\|\|x^{k+1}-u\| \\
+2\langle u, u-x^{k+1}\rangle+\frac{3Q\delta_k}{\theta_k}\|x^k-x^{k-1}\|\Big], \quad \forall k\ge k_0.
\end{aligned}
$$

Setting $t^k=(1-\eta_k)z^k+\eta_k Tz^k$. Using (2.1) and (2.3), we obtain

$$
\begin{aligned}
\|t^k-u\|^2 &= \|(1-\eta_k)(z^k-u)+\eta_k(Tz^k-u)\|^2 \\
&= (1-\eta_k)^2\|z^k-u\|^2+\eta_k^2\|Tz^k-u\|^2 \\
&\quad +2(1-\eta_k)\eta_k\langle Tz^k-u, z^k-u\rangle \\
&\le (1-\eta_k)^2\|z^k-u\|^2+\eta_k^2\|z^k-u\|^2+\eta_k^2\lambda\|Tz^k-z^k\|^2 \\
&\quad +2(1-\eta_k)\eta_k\Big[\|z^k-u\|^2-\frac{1-\lambda}{2}\|Tz^k-z^k\|^2\Big] \\
&= \|z^k-u\|^2+\eta_k[\eta_k-(1-\lambda)]\|Tz^k-z^k\|^2.
\end{aligned} \tag{3.14}
$$

In view of $\{\eta_k\}\subset(0,1-\lambda)$ and (3.12), we get

$$
\|t^k-u\|\le\|s^k-u\|, \quad \forall k\ge k_0. \tag{3.15}
$$

According to the definition of s^k, one obtains

$$
\begin{aligned}
\|s^k-u\|^2 &= \|x^k-u\|^2+2\delta_k\langle x^k-u, x^k-x^{k-1}\rangle+\delta_k^2\|x^k-x^{k-1}\|^2 \\
&\le \|x^k-u\|^2+3Q\delta_k\|x^k-x^{k-1}\|,
\end{aligned} \tag{3.16}
$$

where $Q:=\sup_{k\in\mathbb{N}}\{\|x^k-u\|, \delta\|x^k-x^{k-1}\|\}>0$. Moreover, one sees that

$$\begin{aligned} x^{k+1} &= t^k - \theta_k z^k = (1-\theta_k)t^k - \theta_k(z^k - t^k) \\ &= (1-\theta_k)t^k - \theta_k\eta_k(z^k - Tz^k). \end{aligned}$$

From (3.15) and (3.16), we obtain

$$\begin{aligned} &\|x^{k+1}-u\|^2 \\ &= \|(1-\theta_k)(t^k-u) - \theta_k(\eta_k(z^k-Tz^k)+u)\|^2 \\ &\leq (1-\theta_k)^2\|t^k-u\|^2 - 2\theta_k\langle \eta_k(z^k-Tz^k)+u, x^{k+1}-u\rangle \\ &= (1-\theta_k)^2\|t^k-u\|^2 + \theta_k\left[2\eta_k\langle z^k-Tz^k, u-x^{k+1}\rangle + 2\langle u, u-x^{k+1}\rangle\right] \\ &\leq (1-\theta_k)\|x^k-u\|^2 + \theta_k\left[2\eta_k\|z^k-Tz^k\|\|x^{k+1}-u\| + 2\langle u, u-x^{k+1}\rangle \right. \\ &\quad \left. + \frac{3Q\delta_k}{\theta_k}\|x^k-x^{k-1}\|\right], \quad \forall k \geq k_0. \end{aligned}$$

Claim 4. The sequence $\{\|x^k-u\|^2\}$ converges to zero. We regard to two reasonable situations on the sequence $\{\|x^k-u\|^2\}$.

Case 1: There exists an $N\in\mathbb{N}$ such that $\|x^{k+1}-u\|^2 \leq \|x^k-u\|^2$ for all $k \geq N$. This implies that $\lim_{k\to\infty}\|x^k-u\|^2$ exists. From $\lim_{k\to\infty}\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right) = 1-\phi > 0$ and Claim 2, we obtain

$$\lim_{k\to\infty}\|s^k-y^k\| = 0, \ \lim_{k\to\infty}\|z^k-Tz^k\| = 0, \ \lim_{k\to\infty}\|z^k-y^k\| = 0, \tag{3.17}$$

which implies that $\lim_{k\to\infty}\|z^k-s^k\| = 0$. According to the definition of s^k, one has

$$\|x^k-s^k\| = \delta_k\|x^k-x^{k-1}\| = \theta_k\cdot\frac{\delta_k}{\theta_k}\|x^k-x^{k-1}\| \to 0. \tag{3.18}$$

This together with $\lim_{k\to\infty}\|z^k-s^k\| = 0$ implies that

$$\lim_{k\to\infty}\|z^k-x^k\| = 0. \tag{3.19}$$

Combining Condition (C4), (3.17) and (3.19), we have

$$\|x^{k+1}-x^k\| \leq \|z^k-x^k\| + \theta_k\|z^k\| + \eta_k\|z^k-Tz^k\| \to 0.$$

We suppose that there exists a subsequence $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \rightharpoonup q$ since $\{x^k\}$ is bounded. Moreover,

$$\limsup_{k\to\infty}\langle u, u-x^k\rangle = \lim_{j\to\infty}\langle u, u-x^{k_j}\rangle = \langle u, u-q\rangle.$$

One sees that $s^{k_j} \rightharpoonup q$ because of (3.18), which combining $\lim_{k\to\infty}\gamma_k = \gamma$, (3.17) and Lemma 2.4, yields that $q \in \mathrm{VI}(C,A)$. Furthermore, we get that $z^{k_j} \rightharpoonup q$ by (3.19), which together with $\lim_{k\to\infty}\|z^k-Tz^k\| = 0$ implies that $q \in \mathrm{Fix}(T)$. Thus, we have $q \in \mathrm{VI}(C,A) \cap \mathrm{Fix}(T)$. From $u = P_{\mathrm{VI}(C,A)\cap \mathrm{Fix}(T)}(0)$, one infers that $\limsup_{k\to\infty}\langle u, u - x^k\rangle = \langle u, u-q\rangle \le 0$. By $\|x^{k+1}-x^k\| \to 0$, we obtain

$$\limsup_{k\to\infty}\langle u, u - x^{k+1}\rangle \le 0.$$

Thus, combining Claim 3 and Lemma 2.6, we deduce that $\lim_{k\to\infty}\|x^{k+1}-u\|^2 = 0$. This means that $x^k \to u$ as $k \to \infty$.

Case 2: There is a subsequence $\{\|x^{k_j}-u\|^2\}$ of $\{\|x^k-u\|^2\}$, which, for all $j \in \mathbb{N}$, satisfies $\|x^{k_j}-u\|^2 < \|x^{k_j+1}-u\|^2$. In this situation, according to Lemma 2.5, there is a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$, and the following conclusions hold for all $k \in \mathbb{N}$:

$$\|x^{m_k}-u\|^2 \le \|x^{m_k+1}-u\|^2, \quad \text{and} \quad \|x^k-u\|^2 \le \|x^{m_k+1}-u\|^2. \tag{3.20}$$

From Claim 2, we have

$$\begin{aligned}&\left(1-\phi\frac{\gamma_{m_k}}{\gamma_{m_k+1}}\right)\|y^{m_k}-s^{m_k}\|^2 + \left(1-\phi\frac{\gamma_{m_k}}{\gamma_{m_k+1}}\right)\|z^{m_k}-y^{m_k}\|^2\\&+\eta_{m_k}\left[(1-\lambda)-\eta_{m_k}\right]\|z^{m_k}-Tz^{m_k}\|^2\\&\le \|x^{m_k}-u\|^2 - \|x^{m_k+1}-u\|^2 + \theta_{m_k}Q_4 \le \theta_{m_k}Q_4.\end{aligned}$$

From Condition (C4), we obtain

$$\lim_{k\to\infty}\|s^{m_k}-y^{m_k}\| = \lim_{k\to\infty}\|z^{m_k}-y^{m_k}\| = \lim_{k\to\infty}\|z^{m_k}-Tz^{m_k}\| = 0.$$

As proved in the first situation, we get that $\limsup_{k\to\infty}\langle u, u-x^{m_k+1}\rangle \le 0$. From Claim 3 and (3.20), we obtain

$$\begin{aligned}\|x^{m_k+1}-u\|^2 \le (1-\theta_{m_k})\|x^{m_k+1}-u\|^2 + \theta_{m_k}\big[&2\eta_{m_k}\|z^{m_k}-Tz^{m_k}\|\|x^{m_k+1}-u\|\\&+2\langle u, u-x^{m_k+1}\rangle + \frac{3Q\delta_{m_k}}{\theta_{m_k}}\|x^{m_k}-x^{m_k-1}\|\big],\end{aligned}$$

which implies that

$$\begin{aligned}\|x^k-u\|^2 \le \|x^{m_k+1}-u\|^2 \le 2\eta_{m_k}&\|z^{m_k}-Tz^{m_k}\|\|x^{m_k+1}-u\|\\&+2\langle u, u-x^{m_k+1}\rangle + \frac{3Q\delta_{m_k}}{\theta_{m_k}}\|x^{m_k}-x^{m_k-1}\|.\end{aligned}$$

Thus, $\limsup_{k\to\infty}\|x^k-u\| \le 0$, that is $x^k \to u$ as $k\to\infty$. We have thus proved the theorem. □

3.2 The inertial Mann-type Tseng's extragradient algorithm

In this subsection, we introduce a new iterative scheme which combining the inertial Tseng's extragradient algorithm and the Mann-type method. Note that this method only involves the calculation of one projection in each iteration. Our Algorithm 2 is stated as follows:

Algorithm 2 The inertial Mann-type Tseng's extragradient algorithm

Initialization: Take $\delta > 0$, $\gamma_1 > 0$, $\phi \in (0,1)$. Let $x^0, x^1 \in \mathcal{H}$ be two arbitrary initial points.

Iterative Steps: Calculate the next iteration point x^{k+1} as follows:

$$\begin{cases} s^k = x^k + \delta_k(x^k - x^{k-1}), \\ y^k = P_C(s^k - \gamma_k A s^k), \\ z^k = y^k - \gamma_k(Ay^k - As^k), \\ x^{k+1} = (1 - \theta_k - \eta_k) z^k + \eta_k T z^k, \end{cases}$$

update inertial parameter δ_k and step size γ_{k+1} through (3.1) and (3.2), respectively.

The following lemma is crucial to the proof of the convergence of the algorithm.

Lemma 3.6 *Suppose that Conditions (C1) and (C3) hold. Let the sequence $\{z^k\}$ be generated by Algorithm 2. Then*

$$\|z^k - u\|^2 \le \|s^k - u\|^2 - \left(1 - \phi^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|s^k - y^k\|^2, \quad \forall u \in \mathrm{VI}(C, A),$$

and

$$\|z^k - y^k\| \le \phi \frac{\gamma_k}{\gamma_{k+1}} \|s^k - y^k\|.$$

Proof First, using the definition of γ_k, it is easy to see that

$$\|As^k - Ay^k\| \le \frac{\phi}{\gamma_{k+1}} \|s^k - y^k\|, \quad \forall k \ge 0. \tag{3.21}$$

By the definition of z^k, one sees that

$$
\begin{aligned}
&\|z^k-u\|^2=\|y^k-\gamma_k(Ay^k-As^k)-u\|^2\\
&=\|s^k-u\|^2+\|y^k-s^k\|^2+2\langle y^k-s^k,s^k-u\rangle\\
&\quad+\gamma_k^2\|Ay^k-As^k\|^2-2\gamma_k\langle y^k-u,Ay^k-As^k\rangle\\
&=\|s^k-u\|^2+\|y^k-s^k\|^2-2\langle y^k-s^k,y^k-s^k\rangle+2\langle y^k-s^k,y^k-u\rangle\\
&\quad+\gamma_k^2\|Ay^k-As^k\|^2-2\gamma_k\langle y^k-u,Ay^k-As^k\rangle\\
&=\|s^k-u\|^2-\|y^k-s^k\|^2+2\langle y^k-s^k,y^k-u\rangle\\
&\quad+\gamma_k^2\|Ay^k-As^k\|^2-2\gamma_k\langle y^k-u,Ay^k-As^k\rangle.
\end{aligned}
\tag{3.22}
$$

Combining $y^k=P_C\big(s^k-\gamma_k As^k\big)$ and the property of projection, we obtain

$$\langle y^k-s^k+\gamma_k As^k,y^k-u\rangle\le 0,$$

or equivalently

$$\langle y^k-s^k,y^k-u\rangle\le-\gamma_k\langle As^k,y^k-u\rangle. \tag{3.23}$$

From (3.21), (3.22) and (3.23), we have

$$
\begin{aligned}
\|z^k-u\|^2&\le\|s^k-u\|^2-\|y^k-s^k\|^2-2\gamma_k\langle As^k,y^k-u\rangle\\
&\quad+\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\|s^k-y^k\|^2-2\gamma_k\langle y^k-u,Ay^k-As^k\rangle\\
&=\|s^k-u\|^2-\left(1-\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|s^k-y^k\|^2\\
&\quad-2\gamma_k\langle y^k-u,Ay^k-Au\rangle-2\gamma_k\langle y^k-u,Au\rangle.
\end{aligned}
\tag{3.24}
$$

According to $u\in\mathrm{VI}(C,A)$ and the monotonicity of A, we get

$$\langle Au,y^k-u\rangle\ge 0,\quad\text{and}\quad\langle Ay^k-Au,y^k-u\rangle\ge 0. \tag{3.25}$$

Combining (3.24) and (3.25), we deduce that

$$\|z^k-u\|^2\le\|s^k-u\|^2-\left(1-\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|s^k-y^k\|^2.$$

From the definition of z^k and (3.21), we obtain

$$\|z^k-y^k\|\le\phi\frac{\gamma_k}{\gamma_{k+1}}\|s^k-y^k\|.$$

This completes the proof of the lemma. □

Theorem 3.7 *Suppose that Conditions* $(C1)$–$(C4)$ *hold. Then the sequence* $\{x^k\}$ *created by Algorithm* 2 *converges to* $u \in \text{Fix}(T) \cap \text{VI}(C, A)$ *in norm, where* $\|u\| = \min\{\|p\| : p \in \text{Fix}(T) \cap \text{VI}(C, A)\}$.

Proof By $\lim_{k\to\infty}\left(1-\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\right) = 1-\phi^2 > 0$, one concludes that there exists $k_0 \in \mathbb{N}$ such that

$$1-\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2} > 0, \quad \forall k \geq k_0. \tag{3.26}$$

Combining Lemma 3.6 and (3.26), it follows that

$$\|z^k - u\| \leq \|s^k - u\|, \quad \forall k \geq k_0. \tag{3.27}$$

We also divided the proof into four statements.

Claim 1. The sequence $\{x^k\}$ is bounded. Using the same arguments as in the Claim 1 of Theorem 3.5, we get that $\{x^k\}$ is bounded. So $\{s^k\}$ and $\{z^k\}$ are bounded.

Claim 2.

$$\begin{aligned}&\eta_k[(1-\lambda)-\eta_k]\|z^k - Tz^k\|^2 + \left(1-\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|y^k - s^k\|^2 \\ &\leq \|x^k - u\|^2 - \|x^{k+1} - u\|^2 + \theta_k Q_4.\end{aligned}$$

Indeed, using (3.13) and (3.27) and Lemma 3.6, we obtain

$$\begin{aligned}&\|x^{k+1} - u\|^2 \\ &\leq \|z^k - u\|^2 + \eta_k^2\|Tz^k - z^k\|^2 + 2\eta_k\langle Tz^k - z^k, z^k - u\rangle \\ &\quad + 2\theta_k\langle z^k, u - x^{k+1}\rangle \\ &\leq \|z^k - u\|^2 + \eta_k^2\|Tz^k - z^k\|^2 + \eta_k(\lambda-1)\|z^k - Tz^k\|^2 + \theta_k Q_3 \\ &\leq \|x^k - u\|^2 + \theta_k Q_4 - \eta_k[(1-\lambda)-\eta_k]\|z^k - Tz^k\|^2 \\ &\quad - \left(1-\phi^2\frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|y^k - s^k\|^2,\end{aligned} \tag{3.28}$$

where $Q_4 = Q_2 + Q_3$. Thus, we can obtain the desired result through a direct calculation.

Claim 3.

$$\begin{aligned}\|x^{k+1} - u\|^2 \leq (1-\theta_k)\|x^k - u\|^2 + \theta_k\Big[&2\eta_k\|z^k - Tz^k\|\|x^{k+1} - u\| \\ &+ 2\langle u, u - x^{k+1}\rangle + \frac{3Q\delta_k}{\theta_k}\|x^k - x^{k-1}\|\Big], \quad \forall k \geq k_0.\end{aligned}$$

The desired result can be obtained using the same arguments as in the Claim 3 of

Theorem 3.5.

Claim 4. The sequence $\{\|x^k - u\|\}$ converges to zero. The proof is similar to Claim 4 in Theorem 3.5. We leave it to the reader for confirmation. □

3.3 The modified inertial Mann-type subgradient extragradient algorithm

In this subsection, we present two new modified inertial Mann-type extragradient algorithms to solve fixed point problems and variational inequality problems. First of all, we assume that the next proposed Algorithms 3 and 4 satisfy Conditions (C1)–(C3) and the following Condition (C5).

(C5) Let $\{\zeta_k\}$ and $\{\xi_k\}$ be two nonnegative sequences such that $\lim_{k\to\infty} \frac{\zeta_k}{1-\theta_k} = 0$ and $\sum_{k=1}^{\infty} \xi_k < +\infty$, where $\{\theta_k\} \subset (0,1)$ satisfies $\lim_{k\to\infty}(1-\theta_k) = 0$ and $\sum_{k=1}^{\infty}(1-\theta_k) = \infty$. Let $\{\eta_k\}$ be a real sequence such that $\eta_k \in \left(a, \frac{(1-\lambda)\theta_k}{\lambda+\theta_k}\right) \subset \left(a, \frac{1-\lambda}{1+\lambda}\right) \subset (a, 1-\lambda)$ for some $a > 0$.

Now, we are in a position to show our algorithm, which reads as follows:

Algorithm 3 The modified inertial Mann-type subgradient extragradient algorithm

---

Initialization: Take $\delta > 0$, $\gamma_1 > 0$, $\phi \in (0,1)$. Let $x^0, x^1 \in \mathcal{H}$ be two arbitrary initial points.

Iterative Steps: Calculate the next iteration point x^{k+1} as follows:

$$\begin{cases} s^k = x^k + \delta_k(x^k - x^{k-1}), \\ y^k = P_C(s^k - \gamma_k A s^k), \\ H_k = \{x \in \mathcal{H} \mid \langle s^k - \gamma_k A s^k - y^k, x - y^k\rangle \le 0\}, \\ z^k = P_{H_k}(s^k - \gamma_k A y^k), \\ x^{k+1} = (1-\eta_k)(\theta_k z^k) + \eta_k T z^k, \end{cases}$$

update inertial parameter δ_k and step size γ_{k+1} through (3.1) and (3.2), respectively.

Theorem 3.8 *Suppose that Conditions* (C1)–(C3) *and* (C5) *hold. Then the iterative sequence* $\{x^k\}$ *formed by Algorithm* 3 *converges to* $u \in \operatorname{Fix}(T) \cap \mathrm{VI}(C,A)$ *in norm, where* $\|u\| = \min\{\|p\| : p \in \operatorname{Fix}(T) \cap \mathrm{VI}(C,A)\}$.

Proof We also divide the proof into four claims.

Claim 1. The sequence $\{x^k\}$ is bounded. From the definition of s^k, one has

$$\begin{aligned} \|s^k - u\| &= \|x^k + \delta_k(x^k - x^{k-1}) - u\| \\ &\le \|x^k - u\| + (1-\theta_k) \cdot \frac{\delta_k}{1-\theta_k} \|x^k - x^{k-1}\|. \end{aligned} \tag{3.29}$$

According to Condition (C5), we have $\frac{\delta_k}{1-\theta_k}\|x^k - x^{k-1}\| \to 0$ as $k \to \infty$. Thus, there is a constant $Q_1 > 0$ such that

$$\frac{\delta_k}{1-\theta_k}\|x^k - x^{k-1}\| \le Q_1, \quad \forall k \ge 1. \tag{3.30}$$

Combining (3.7), (3.29) and (3.30), we find that

$$\|z^k - u\| \le \|s^k - u\| \le \|x^k - u\| + (1-\theta_k)Q_1, \quad \forall k \ge k_0. \tag{3.31}$$

Furthermore, by the definition of x^{k+1}, one obtains

$$\begin{aligned} &\|x^{k+1} - u\| \\ &= \|\theta_k(1-\eta_k)(z^k - u) + \eta_k(Tz^k - u) - (1-\eta_k)(1-\theta_k)u\| \\ &\le \|\theta_k(1-\eta_k)(z^k - u) + \eta_k(Tz^k - u)\| + (1-\eta_k)(1-\theta_k)\|u\|. \end{aligned} \tag{3.32}$$

Since $\eta_k < \frac{(1-\lambda)\theta_k}{\lambda+\theta_k}$, one infers that

$$\lambda\eta_k < (1-\lambda)\theta_k - \theta_k\eta_k < \theta_k(1-\lambda)(1-\eta_k).$$

From (2.1) and (2.3), we get

$$\begin{aligned} &\|\theta_k(1-\eta_k)(z^k - u) + \eta_k(Tz^k - u)\|^2 \\ &= (\theta_k(1-\eta_k))^2\|z^k - u\|^2 + \eta_k^2\|Tz^k - u\|^2 \\ &\quad + 2\theta_k(1-\eta_k)\eta_k\langle Tz^k - u, z^k - u\rangle \\ &\le (\theta_k(1-\eta_k))^2\|z^k - u\|^2 + \eta_k^2\|z^k - u\|^2 + \eta_k^2\lambda\|Tz^k - z^k\|^2 \\ &\quad + 2\theta_k(1-\eta_k)\eta_k\|z^k - u\|^2 - \theta_k(1-\lambda)(1-\eta_k)\eta_k\|Tz^k - z^k\|^2 \\ &= (\theta_k(1-\eta_k) + \eta_k)^2\|z^k - u\|^2 \\ &\quad + \eta_k(\lambda\eta_k - \theta_k(1-\lambda)(1-\eta_k))\|Tz^k - z^k\|^2 \\ &\le (\theta_k(1-\eta_k) + \eta_k)^2\|z^k - u\|^2, \end{aligned} \tag{3.33}$$

which combining with (3.31) further yields that

$$
\begin{aligned}
&\|\theta_k(1-\eta_k)(z^k-u)+\eta_k(Tz^k-u)\| \\
&\leq (\theta_k(1-\eta_k)+\eta_k)\|z^k-u\| \\
&\leq (1-(1-\eta_k)(1-\theta_k))\|x^k-u\|+(1-\theta_k)Q_1.
\end{aligned} \tag{3.34}
$$

Combining (3.32) and (3.34), we have

$$
\begin{aligned}
\|x^{k+1}-u\| &\leq (1-(1-\eta_k)(1-\theta_k))\|x^k-u\| \\
&\quad + (1-\eta_k)(1-\theta_k)\left[\|u\|+\frac{Q_1}{1-\eta_k}\right] \\
&\leq \max\left\{\|x^k-u\|, \|u\|+\frac{Q_1}{1-\eta_k}\right\} \\
&\leq \cdots \leq \max\left\{\|x^{k_0}-u\|, \|u\|+\frac{Q_1}{1-\eta_k}\right\}.
\end{aligned}
$$

Consequently, the sequence $\{x^k\}$ is bounded. So the sequences $\{s^k\}$ and $\{z^k\}$ are also bounded.

Claim 2.

$$
\begin{aligned}
&\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|y^k-s^k\|^2+\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|z^k-y^k\|^2 \\
&+\eta_k(1-\lambda-\eta_k)\|Tz^k-z^k\|^2 \\
&\leq \|x^k-u\|^2-\|x^{k+1}-u\|^2+(1-\theta_k)Q_4.
\end{aligned}
$$

Indeed, it follows from (3.31) that

$$
\begin{aligned}
\|s^k-u\|^2 &\leq (\|x^k-u\|+(1-\theta_k)Q_1)^2 \\
&= \|x^k-u\|^2+(1-\theta_k)(2Q_1\|x^k-u\|+(1-\theta_k)Q_1^2) \\
&\leq \|x^k-u\|^2+(1-\theta_k)Q_2
\end{aligned} \tag{3.35}
$$

for some $Q_2>0$. Using (2.2), (3.31), (3.35) and Lemma 3.4, we obtain

$$\begin{aligned}\|x^{k+1}-u\|^2 &= \|(z^k-u)+\eta_k(Tz^k-z^k)-(1-\eta_k)(1-\theta_k)z^k\|^2 \\ &\le \|z^k-u\|^2+\eta_k^2\|Tz^k-z^k\|^2+2\eta_k\langle Tz^k-z^k, z^k-u\rangle \\ &\quad -2(1-\eta_k)(1-\theta_k)\langle z^k, x^{k+1}-u\rangle \\ &\le \|z^k-u\|^2+\eta_k^2\|Tz^k-z^k\|^2-\eta_k(1-\lambda)\|Tz^k-z^k\|^2 \\ &\quad +2(1-\eta_k)(1-\theta_k)\langle z^k, u-x^{k+1}\rangle \\ &\le \|z^k-u\|^2-\eta_k(1-\lambda-\eta_k)\|Tz^k-z^k\|^2+(1-\theta_k)Q_3 \\ &\le \|x^k-u\|^2-\eta_k(1-\lambda-\eta_k)\|Tz^k-z^k\|^2+(1-\theta_k)Q_4 \\ &\quad -\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|y^k-s^k\|^2-\left(1-\phi\frac{\gamma_k}{\gamma_{k+1}}\right)\|z^k-y^k\|^2,\end{aligned}$$

where $Q_4 = Q_2 + Q_3$. Thus, we can obtain the desired result through a direct calculation.

Claim 3.

$$\begin{aligned}&\|x^{k+1}-u\|^2 \\ &\le [1-(1-\eta_k)(1-\theta_k)]\|x^k-u\|^2+(1-\eta_k)(1-\theta_k)\Big[2\langle u, u-x^{k+1}\rangle \\ &\quad +2\eta_k\|Tz^k-z^k\|\|x^{k+1}-u\|+\frac{3Q}{(1-\eta_k)}\cdot\frac{\delta_k}{(1-\theta_k)}\|x^k-x^{k-1}\|\Big].\end{aligned}$$

Taking $t^k = (1-\eta_k)z^k+\eta_k Tz^k$, then as proved in Claim 3 of Theorem 3.5, we get that $\|t^k-u\| \le \|s^k-u\|$. This together with (3.16) yields that

$$\begin{aligned}&\|x^{k+1}-u\|^2 = \|t^k-u-(1-\eta_k)(1-\theta_k)z^k\|^2 \\ &= \|[1-(1-\eta_k)(1-\theta_k)](t^k-u)+(1-\eta_k)(1-\theta_k)[(t^k-z^k)-u]\|^2 \\ &= \|[1-(1-\eta_k)(1-\theta_k)](t^k-u)+(1-\eta_k)(1-\theta_k)[\eta_k(Tz^k-z^k)-u]\|^2 \\ &\le [1-(1-\eta_k)(1-\theta_k)]^2\|t^k-u\|^2 \\ &\quad +2(1-\eta_k)(1-\theta_k)\langle \eta_k(Tz^k-z^k)-u, x^{k+1}-u\rangle \\ &\le [1-(1-\eta_k)(1-\theta_k)]\|x^k-u\|^2+(1-\eta_k)(1-\theta_k)\Big[2\langle u, u-x^{k+1}\rangle \\ &\quad +2\eta_k\|Tz^k-z^k\|\|x^{k+1}-u\|+\frac{3Q}{(1-\eta_k)}\cdot\frac{\delta_k}{(1-\theta_k)}\|x^k-x^{k-1}\|\Big].\end{aligned}$$

Claim 4. The sequence $\{\|x^k-u\|\}$ converges to zero. The proof of this result is similar to that of Theorem 3.5. We leave it to the reader for confirmation. □

3.4 The modified inertial Mann-type Tseng's extragradient algorithm

Finally, we introduce a modified inertial Mann-type Tseng's extragradient algorithm. The details of the Algorithm 4 are described as follows.

Algorithm 4 The modified inertial Mann-type Tseng's extragradient algorithm

---

Initialization: Take $\delta > 0$, $\gamma_1 > 0$, $\phi \in (0,1)$. Let $x^0, x^1 \in \mathcal{H}$ be two arbitrary initial points.

Iterative Steps: Calculate the next iteration point x^{k+1} as follows:

$$\begin{cases} s^k = x^k + \delta_k(x^k - x^{k-1}) \,, \\ y^k = P_C(s^k - \gamma_k A s^k) \,, \\ z^k = y^k - \gamma_k(Ay^k - As^k) \,, \\ x^{k+1} = (1-\eta_k)(\theta_k z^k) + \eta_k T z^k \,, \end{cases}$$

update inertial parameter δ_k and step size γ_{k+1} through (3.1) and (3.2), respectively.

Theorem 3.9 *Suppose that Conditions* $(C1)$–$(C3)$ *and* $(C5)$ *hold. Then the iterative sequence* $\{x^k\}$ *created by Algorithm* 4 *converges to* $u \in \text{Fix}\,(T) \cap \text{VI}(C,A)$ *in norm, where* $\|u\| = \min\{\|p\| : p \in \text{Fix}\,(T) \cap \text{VI}(C,A)\}$.

Proof The proof is divided into four steps.

Claim 1. The sequence $\{x^k\}$ is bounded. As proved in Theorem 3.7, we also get that $\|z^k - u\| \le \|s^k - u\|, \forall k \ge k_0$. Using the same arguments as in Claim 1 of Theorem 3.8, one concludes that $\{x^k\}$ is bounded. So $\{s^k\}$ and $\{z^k\}$ are bounded.

Claim 2.

$$\begin{aligned} &\eta_k(1-\lambda-\eta_k)\|Tz^k - z^k\|^2 + \left(1 - \phi^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|y^k - s^k\|^2 \\ &\le \|x^k - u\|^2 - \|x^{k+1} - u\|^2 + (1-\theta_k)Q_4. \end{aligned}$$

Indeed, using (3.35) and Lemma 3.6, we have

$$\begin{aligned} \|x^{k+1} - u\|^2 &\le \|z^k - u\|^2 + \eta_k^2\|Tz^k - z^k\|^2 - \eta_k(1-\lambda)\|Tz^k - z^k\|^2 \\ &\quad + 2(1-\eta_k)(1-\theta_k)\langle z^k, u - x^{k+1}\rangle \\ &\le \|z^k - u\|^2 - \eta_k(1-\lambda-\eta_k)\|Tz^k - z^k\|^2 + (1-\theta_k)Q_3 \\ &\le \|x^k - u\|^2 - \eta_k(1-\lambda-\eta_k)\|Tz^k - z^k\|^2 + (1-\theta_k)Q_4 \\ &\quad - \left(1 - \phi^2 \frac{\gamma_k^2}{\gamma_{k+1}^2}\right)\|y^k - s^k\|^2, \end{aligned}$$

where $Q_4 = Q_2 + Q_3$. Thus, we can obtain the desired result through a direct calculation.

Claim 3.

$$
\begin{aligned}
&\|x^{k+1}-u\|^2\\
&\le[1-(1-\eta_k)(1-\theta_k)]\|x^k-u\|^2+(1-\eta_k)(1-\theta_k)\Big[2\langle u,u-x^{k+1}\rangle\\
&\quad+2\eta_k\|Tz^k-z^k\|\|x^{k+1}-u\|+\frac{3Q}{(1-\eta_k)}\cdot\frac{\delta_k}{(1-\theta_k)}\|x^k-x^{k-1}\|\Big].
\end{aligned}
$$

The desired result can be obtained by using the same arguments as in the Claim 3 of Theorem 3.8.

Claim 4. The sequence $\{\|x^k-u\|\}$ converges to zero. The proof is similar to Claim 4 in Theorem 3.5. We leave it to the reader for confirmation. □

4 Numerical examples

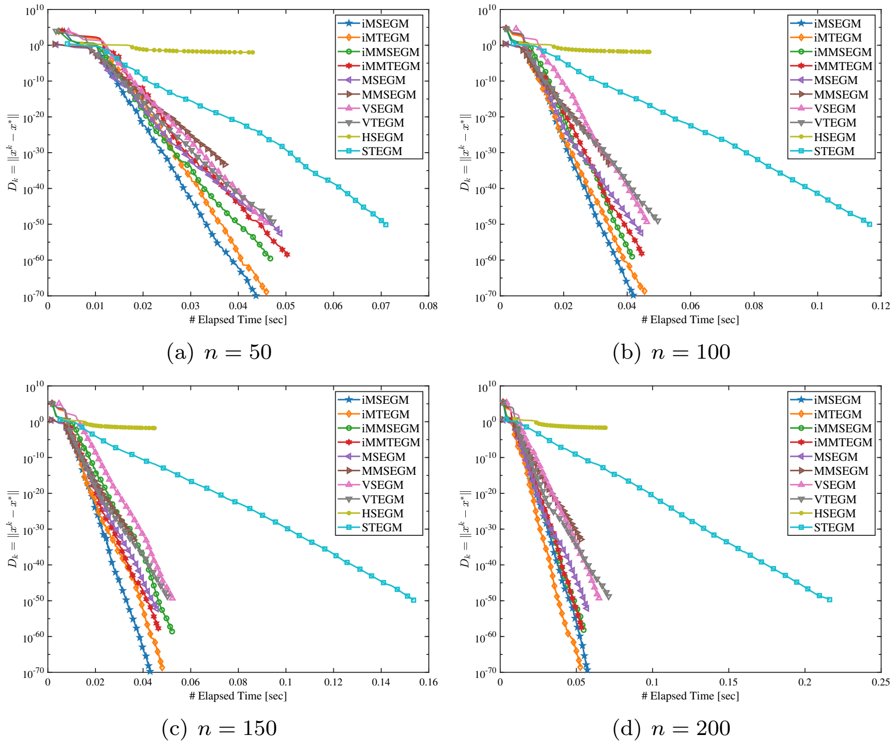
In this section, we provide several computational tests to illustrate the numerical behavior of our proposed algorithms (For convenience, we abbreviate Algorithm 1 as iMSEGM, Algorithm 2 as iMTEGM, Algorithm 3 as iMMSEGM and Algorithm 4 as iMMTEGM), and compare them with some existing strongly convergent methods, which including the Halpern subgradient extragradient method (HSEGM) [12], the self adaptive Tseng's extragradient method (STEGM) [24], the Mann-type subgradient extragradient method (MSEGM) [25], the modified Mann-type subgradient extragradient method (MMSEGM) [25], the Viscosity-type subgradient extragradient method (VSEGM) [26] and the Viscosity-type Tseng's extragradient method (VTEGM) [26].

Table 1 Parameter setting for all algorithms

Algorithms	Parameters
HSEGM	$\theta_k=1/(k+1)$, $\eta_k=k/(2k+1)$, $\gamma=0.99/L$.
MSEGM	$\theta_k=1/(k+1)$, $\eta_k=0.5(1-\theta_k)$, $\gamma=0.99/L$.
MMSEGM	$\theta_k=n/(n+1)$, $\eta_k=\theta_k/3$, $\gamma=0.99/L$.
iMSEGM	$\theta_k=1/(k+1)$, $\eta_k=0.5(1-\theta_k)$, $\delta=0.6$, $\zeta_k=1/(k+1)^2$, $\phi=0.5$, $\gamma_1=0.5$, $\xi_k=1/(k+1)^{1.1}$.
iMTEGM	The parameters set are the same as Algorithm (iMSEGM).
iMMSEGM	$\theta_k=k/(k+1)$, $\eta_k=\theta_k/3$, $\delta=0.6$, $\zeta_k=1/(k+1)^2$, $\phi=0.5$, $\gamma_1=0.5$, $\xi_k=1/(k+1)^{1.1}$.
iMTSEGM	The parameters set are the same as Algorithm (iMMSEGM).
VSEGM	$\theta_k=1/(k+1)$, $\eta_k=k/(2k+1)$, $\phi=0.5$, $\gamma_1=0.5, f(x)=0.5x$.
VTEGM	The parameters set are the same as Algorithm (VSEGM).
STEGM	$\theta_k=1/(k+1)$, $\eta_k=k/(2k+1)$, $\rho=1$, $l=0.5$, $\phi=0.4$, $\lambda=0.5$.

The parameters of all the algorithms are set in Table 1. In our experiment examples, the solution x^* of the problems are known. Therefore, we take $D_k = \|x^k - x^*\|$ to evaluate the kth iteration error. Note that the sequence $\{D_k\} \to 0$ implies that $\{x^k\}$ converges to the solution of the problem. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

Example 4.1 Consider the form of linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ ($n = 50, 100, 150, 200$) as follows: $A(x) = Gx + f$, where $f \in \mathbb{R}^n$ and $G = BB^\mathsf{T} + S + E$, matrix $B \in \mathbb{R}^{n\times n}$, matrix $S \in \mathbb{R}^{n\times n}$ is skew-symmetric, and matrix $E \in \mathbb{R}^{n\times n}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set as $C = \{x \in \mathbb{R}^n : -2 \le x_i \le 5,\ i = 1, \ldots, n\}$. It is easy to see that A is Lipschitz continuous and monotone, and its Lipschitz constant $L = \|G\|$. In this numerical example, both B, E entries are randomly created in [0, 2], S is generated randomly in $[-2, 2]$ and $f = \mathbf{0}$. Let $T : \mathcal{H} \to \mathcal{H}$ and $F : \mathcal{H} \to \mathcal{H}$ be provided by $Tx = 0.5x$ and $Fx = 0.5x$, respectively. We obtain the solution of the problem is $x^* = \{\mathbf{0}\}$. The maximum number of iterations 400 as a common stopping criterion for all algorithms and the initial values $x^0 = x^1$ are randomly generated by *rand(2,1)* in



(a) $n = 50$ (b) $n = 100$

(c) $n = 150$ (d) $n = 200$

Fig. 1 Numerical results of all algorithms for Example 4.1

MATLAB. The numerical behaviors of all algorithms with execution time in four dimensions are shown in Fig. 1.

Example 4.2 In this numerical example, we focus on a case in Hilbert space $\mathcal{H} = L^2([0,1])$. Its inner product and induced norm are defined as $\langle m, n\rangle := \int_0^1 m(t)n(t)\,\mathrm{d}t$ and $\|m\| := (\int_0^1 |m(t)|^2\,\mathrm{d}t)^{1/2}$, respectively. Let the feasible set C be given by $C = \{x \in \mathcal{H} \mid \|x\| \le 1\}$. Assume that the operator $A : C \to \mathcal{H}$ is defined as follows:

$$(Ax)(t) = \max\{x(t), 0\} = \frac{x(t) + |x(t)|}{2}.$$

It is easy to verify that A is monotone and 1-Lipschitz continuous, and the projection on C is inherently explicit, that is,

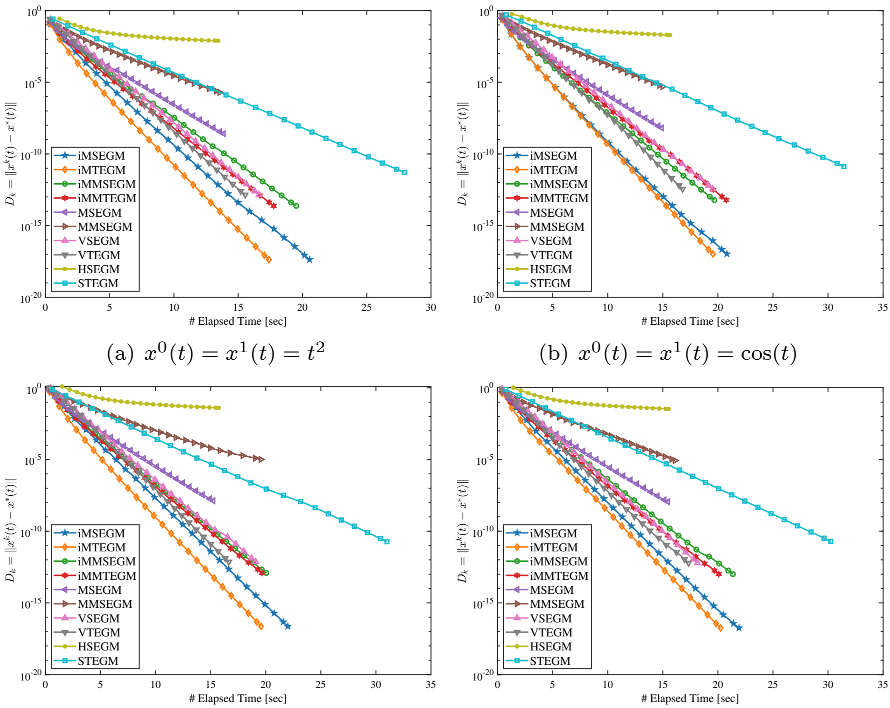
$$P_C(x) = \begin{cases} x, & \text{if } \|x\| \le 1; \\ \dfrac{x}{\|x\|}, & \text{if } \|x\| > 1. \end{cases}$$

The mapping $T : L^2([0,1]) \to L^2([0,1])$ is of the form,

$$(Tx)(t) = \int_0^1 tx(r)\,\mathrm{d}r, \quad t \in [0,1].$$

A simple computation indicates that T is 0-demicontractive and demiclosed at 0. Let operator $F : \mathcal{H} \to \mathcal{H}$ be defined as $(Fx)(t) = 0.5x(t)$. It is easy to check that operator F is Lipschitz continuous and strongly monotone. Through a straightforward calculation, we know that the solution of the problem is $x^*(t) = 0$. The maximum number of iterations 50 as a common stopping criterion for all algorithms. With four types of initial points $x^0(t) = x^1(t)$, the numerical behaviors of function $D_k = \|x^k(t) - x^*(t)\|$ with elapsed time are described in Fig. 2.

Remark 4.3 From the above numerical examples appearing in finite- and infinite-dimensional spaces, it can be seen that the proposed algorithms have a higher convergence accuracy under the same stopping conditions. The convergence speed of our algorithms is faster than that of some known algorithms in the literature [12, 24–26], and these results are independent of the size of dimensions and the selection of initial values. More importantly, the algorithms obtained in this paper automatically updates the step size through a simple calculation, which makes the suggested algorithms work well without the prior information of the Lipschitz constant of the mapping.



(a) $x^0(t) = x^1(t) = t^2$ (b) $x^0(t) = x^1(t) = \cos(t)$

(c) $x^0(t) = x^1(t) = e^t$ (d) $x^0(t) = x^1(t) = 2^t$

Fig. 2 Numerical results of all algorithms for Example 4.2

5 Final remarks

In this research, we presented four new inertial extragradient algorithms with a new non-monotonic step size for seeking common solutions of a monotone variational inequality problem and a fixed point problem in a real Hilbert space. The advantage of the proposed algorithms is that we do not need to know the prior information of the Lipschitz constant of the mapping in advance. In addition, our algorithms add an inertial term, which significantly improves the convergence speed of our algorithms. Strong convergence of the suggested algorithms were proved under certain suitable conditions imposed on parameters. Some numerical examples were presented to demonstrate the performance of the suggested algorithms over some previously known ones. The iterative schemes obtained in this paper improved and extended some results in the literature.

Acknowledgements The authors are very grateful to the Associate Editor and the anonymous referee for their valuable suggestions, which helped us to improve the initial manuscript.

Declaration

Conflict of interest The authors declare that there is no conflict of interest.

References

1. Ansari, Q.H., Islam, M., Yao, J.C.: Nonsmooth variational inequalities on Hadamard manifolds. Appl. Anal. **99**, 340–358 (2020)
2. Censor, Y., Gibali, A., Reich, S.: Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space. Optim. Methods Softw. **26**, 827–845 (2011)
3. Ceng, L.C., Petruşel, A., Yao, J.C., Yao, Y.: Hybrid viscosity extragradient method for systems of variational inequalities, fixed points of nonexpansive mappings, zero points of accretive operators in Banach spaces. Fixed Point Theory **19**, 487–502 (2018)
4. Dong, Q.L., He, S., Rassias, M.T.: General splitting methods with linearization for the split feasibility problem. J. Global Optim. **79**, 813–836 (2021)
5. Dong, Q.L., Cho, Y.J., Zhong, L.L., Rassias, T.M.: Inertial projection and contraction algorithms for variational inequalities. J. Global Optim. **70**, 687–704 (2018)
6. Gibali, A., Thong, D.V.: A new low-cost double projection method for solving variational inequalities. Optim. Eng. **21**, 1613–1634 (2021)
7. Gibali, A., Shehu, Y.: An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces. Optimization **68**, 13–32 (2019)
8. Gibali, A., Hieu D.V.: A new inertial double-projection method for solving variational inequalities. J. Fixed Point Theory Appl. **21**, Article ID 97 (2019)
9. Hieu, D.V., Gibali, A.: Strong convergence of inertial algorithms for solving equilibrium problems. Optim. Lett. **14**, 1817–1843 (2020)
10. Khan, A.A., Motreanu, D.: Inverse problems for quasi-variational inequalities. J. Global Optim. **70**, 401–411 (2018)
11. Korpelevich, G.M.: The extragradient method for finding saddle points and other problems. Èkon. i Mat. Metody **12**, 747–756 (1976)
12. Kraikaew, R., Saejung, S.: Strong convergence of the Halpern subgradient extragradient method for solving variational inequalities in Hilbert spaces. J. Optim. Theory Appl. **163**, 399–412 (2014)
13. Liu, H., Yang, J.: Weak convergence of iterative methods for solving quasimonotone variational inequalities. Comput. Optim. Appl. **77**, 491–508 (2020)
14. Maingé, P.E.: A hybrid extragradient-viscosity method for monotone operators and fixed point problems. SIAM J. Control. Optim. **47**, 1499–1515 (2008)
15. Nam, N.M., Rector, R.B., Giles, D.: Minimizing differences of convex functions with applications to facility location and clustering. J. Optim. Theory Appl. **173**, 255–278 (2017)
16. Sahu, D.R., Yao, J.C., Verma, M., Shukla, K.K.: Convergence rate analysis of proximal gradient methods with applications to composite minimization problems. Optimization **70**, 75–100 (2021)
17. Shehu, Y., Li, X.H., Dong, Q.L.: An efficient projection-type method for monotone variational inequalities in Hilbert spaces. Numer. Algorithms **84**, 365–388 (2020)
18. Shehu, Y., Iyiola, O.S.: Iterative algorithms for solving fixed point problems and variational inequalities with uniformly continuous monotone operators. Numer. Algorithms **79**, 529–553 (2018)
19. Shehu, Y., Iyiola, O.S.: Strong convergence result for monotone variational inequalities. Numer. Algorithms **76**, 259–282 (2017)
20. Shehu, Y., Iyiola, O.S., Reich, S.: A modified inertial subgradient extragradient method for solving variational inequalities. Optim. Eng. https://doi.org/10.1007/s11081-020-09593-w (2021)
21. Shehu, Y., Iyiola, O.S.: Projection methods with alternating inertial steps for variational inequalities: weak and linear convergence. Appl. Numer. Math. **157**, 315–337 (2020)
22. Shehu, Y., Liu, L., Mu, X., Dong, Q.L.: Analysis of versions of relaxed inertial projection and contraction method. Appl. Numer. Math. **165**, 1–21 (2021)
23. Tseng, P.: A modified forward-backward splitting method for maximal monotone mappings. SIAM J. Control. Optim. **38**, 431–446 (2000)
24. Tong, M.Y., Tian, M.: Strong convergence of the Tseng extragradient method for solving variational inequalities. Appl. Set-Valued Anal. Optim. **2**, 19–33 (2020)
25. Thong, D.V., Hieu, D.V.: Modified subgradient extragradient algorithms for variational inequality problems and fixed point problems. Optimization **67**, 83–102 (2018)
26. Thong, D.V., Hieu, D.V.: Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems. Numer. Algorithms **82**, 761–789 (2019)
27. Tan, B., Fan, J., Li, S.: Self-adaptive inertial extragradient algorithms for solving variational inequality problems. Comput. Appl. Math. **40**, Article ID 19 (2021)

28. Xu, H.K.: Iterative algorithms for nonlinear operators. J. Lond. Math. Soc. **66**, 240–256 (2002)
29. Zhao, X., Yao, Y.: Modified extragradient algorithms for solving monotone variational inequalities and fixed point problems. Optimization **69**, 1987–2002 (2020)
30. Zhou, Z., Tan, B., Li, S.: A new accelerated self-adaptive stepsize algorithm with excellent stability for split common fixed point problems. Comput. Appl. Math. **39**, Article ID 220 (2020)