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Self-adaptive inertial extragradient algorithms for solving variational inequality problems

Bing Tan[1] · Jingjing Fan[1] · Songxiao Li[1]

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Abstract
In this paper, we study the strong convergence of two Mann-type inertial extragradient algorithms, which are devised with a new step size, for solving a variational inequality problem with a monotone and Lipschitz continuous operator in real Hilbert spaces. Strong convergence theorems for the suggested algorithms are proved without the prior knowledge of the Lipschitz constant of the operator. Finally, we provide some numerical experiments to illustrate the performance of the proposed algorithms and provide a comparison with related ones.

Keywords Variational inequality problem · Subgradient extragradient algorithm · Tseng's extragradient algorithm · Inertial method · Mann-type method

Mathematics Subject Classification 47H05 · 47H09 · 47J20 · 47J25 · 65K10 · 65K15

1 Introduction

Let C be a convex and closed set in a real Hilbert spaces $\mathscr{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. For all $x, y \in \mathscr{H}$, one recalls that a mapping $T : \mathscr{H} \to \mathscr{H}$ is said to be (i) L-Lipschitz continuous with $L > 0$ iff $\|Tx - Ty\| \leq L\|x - y\|$ (if $L = 1$, then T is said to be nonexpansive); (ii) η-strongly monotone if there exists $\eta > 0$ such that $\langle Tx - Ty, x - y \rangle \geq \eta \|x - y\|$; (iii) monotone if $\langle Tx - Ty, x - y \rangle \geq 0$. A point $x^* \in \mathscr{H}$ is called a fixed point of T if $Tx^* = x^*$. The set of all the fixed points of T is denoted by

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✉ Songxiao Li
jyulsx@163.com

Bing Tan
bingtan72@gmail.com

Jingjing Fan
fanjingjing0324@163.com

1 Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, China

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$\mathrm{Fix}(T)$. Let $A : \mathscr{H} \to \mathscr{H}$ be an operator. The variational inequality problem (shortly, VIP) for A on C is to find a point $x^* \in C$ such that

$$\left\langle Ax^*, x - x^* \right\rangle \geq 0, \quad \forall x \in C \,. \tag{VIP}$$

From now on, the solution set of (VIP) is denoted by $\mathrm{VI}(C, A)$.

In a wide range of applied mathematics problems, the existence of a solution is equivalent to the solution of the above-mentioned classical variational inequality. Therefore, variational inequalities are important tools for studying various physics, engineering, economics and optimization theories, see, e.g., (Wang et al. 2019; Sahu et al. 2020; Qin and An 2019; An et al. 2020). Over the last 60 years or so, the variational inequality has been revealed as a very powerful and important tool in the study of various linear and nonlinear phenomena. Some problems, such as systems of equations, complementarity problems, and equilibrium problems, can be formulated as variational inequalities.

Recently, many authors proposed and investigated various algorithms for solving the variational inequality problem, see, e.g., (Cho and Kang 2012; Cho et al. 2013; Shehu et al. 2019; Liu et al. 2019; Fan et al. 2020; Ansari et al. 2020; Wang and Pham 2019) and the references therein. Projection-based methods and their variant forms act as important tools for finding approximate solutions of the variational inequality. A well-known method to solve (VIP) is the projected gradient method: $x_{n+1} = P_C\left(x_n - \lambda A x_n\right), \forall n \geq 1$, where λ is a positive real number and P_C is the metric (nearest point) projection onto C. However, the convergence of the algorithm requires strong monotonicity of A (or inverse strongly, which is also usually said to be co-coercive). If mapping A is L-Lipschitz continuous and monotone, Korpelevich (1976) proposed the following extragradient method (EGM) with double projections to reduce the strong hypotheses of operator A:

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right)\,, \\ x_{n+1} = P_C\left(x_n - \lambda A y_n\right), \quad \forall n \geq 1\,, \end{cases}$$

where $\lambda \in (0, 1/L)$. The algorithm converges to an element of $\mathrm{VI}(C, A)$ provided that $\mathrm{VI}(C, A)$ is non-empty. The disadvantage of the EGM is that it needs to calculate two projections from $\mathscr{H}$ onto the feasibility set C in each iteration. If C is a general convex-closed set, this might require a prohibitive amount of computation time. To overcome this computational drawback, many authors have modified this method in various ways. Recently, there are two modified extragradient algorithms in the literature to overcome this shortcoming. These two methods are the Tseng's extragradient algorithm (TEGM) suggested by Tseng (2000) and the subgradient extragradient algorithm (SEGM) proposed by Censor et al. (2011). We point out here that the Tseng's extragradient algorithm and the subgradient extragradient algorithm only need to calculate one projection onto C in each iteration. Note that under some appropriate settings, the TEGM and the SEGM weakly converge to the solution of the variational inequality. Some examples in machine learning and image processing tell us that strong convergence is preferable to weak convergence in an infinite-dimensional space. For this reason, a natural question is how to design an algorithm that provides strong convergence to solve the (VIP) when mapping A is only L-Lipschitz continuous and monotone. Recently, Kraikaew and Saejung (2014) based on the subgradient extragradient algorithm and the Halpern method to proposed an algorithm for solving monotone (VIP). Their algorithm is of the form:

$$\begin{cases} y_n = P_C\left(x_n - \lambda A x_n\right) , \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \lambda A x_n - y_n, x - y_n\rangle \leq 0\} , \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\, P_{T_n}\left(x_n - \lambda A y_n\right) , \quad \forall n \geq 1 , \end{cases} \tag{HSEGM}$$

where $\lambda \in (0, 1/L)$, $\alpha_n \subset (0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and $\lim_{n\to\infty} \alpha_n = 0$. They proved that the sequence $\{x_n\}$ generated by (HSEGM) converges to the solution of (VIP) in norm. Note that the algorithm (HSEGM) needs to know the Lipschitz constant of the mapping A, which limits the applicability of the algorithm. To overcome this shortcoming, Shehu and Iyiola (2017) proposed a modification of the subgradient extragradient algorithm with the adoption of the Armijo-like step size rule. Indeed, they investigated the following algorithm:

$$\begin{cases} y_n = P_C\left(x_n - \lambda_n A x_n\right) , \\ T_n = \{x \in \mathscr{H} \mid \langle x_n - \lambda_n A x_n - y_n, x - y_n\rangle \leq 0\} , \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) P_{T_n}\left(x_n - \lambda_n A y_n\right) , \quad \forall n \geq 1 , \end{cases} \tag{VSEGM}$$

where $f : \mathscr{H} \to \mathscr{H}$ is a contraction mapping, $\lambda_n = \ell^{m_n}$ and m_n is the smallest non-negative integer such that $\lambda_n \|Ax_n - Ay_n\| \leq \mu \|x_n - y_n\|$ ($\ell \in (0,1)$, $\mu \in (0,1)$). They showed that the iterative process defined by (VSEGM) converges to the solution set of (VIP) in norm. The algorithm does not need to know the Lipschitz constant of the mapping A, but calculating the step size requires to evaluate the value of A multiple times in each iteration. Therefore, although the Armijo-like criterion may not need to know the Lipschitz constant, it is very computationally expensive. Recently, Yang and Liu (2019) combined the Tseng's extragradient algorithm and the viscosity method with a simple step size and proposed a new iterative algorithm. The algorithm consists of only one projection and does not require the prior knowledge of the Lipschitz constant of the operator. They obtained a strong convergence theorem under suitable conditions, and their algorithm is described as follows:

$$\begin{cases} \text{Take } \lambda_0 \in (0,1),\ \mu \in (0,1) , \\ y_n = P_C\left(x_n - \lambda_n A x_n\right) , \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)[y_n - \lambda_n (A y_n - A x_n)] , \\ \lambda_{n+1} = \begin{cases} \min\left\{ \frac{\mu \|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \lambda_n \right\}, & \text{if } Ax_n - Ay_n \neq 0 ; \\ \lambda_n , & \text{otherwise} . \end{cases} \end{cases} \tag{TVEGM}$$

On the other hand, in recent years, there has been tremendous interest in developing fast iterative algorithms. Many authors have used inertial methods to devise a large number of iterative algorithms that can improve the convergence speed; see, for example, (Liu 2019; Qin et al. 2020; Tan et al. 2020; Tan and Li 2020; Tan and Xu 2020; Zhou et al. 2020) and the references therein. These inertial-type algorithms have better numerical performance than algorithms without inertial terms.

Motivated and inspired by the above works, in this paper, we introduce two self-adaptive inertial Mann-type extragradient algorithms to solve the monotone variational inequality problem in real Hilbert spaces. Our algorithms can work well without knowing the prior knowledge of the Lipschitz constant of the mapping. Under some mild conditions, we prove that the iterative sequence generated by the suggested algorithms converges to a solution of (VIP) in norm. Some numerical experiments are provided to support the theoretical results. Our numerical results show that the new algorithms have a faster convergence speed than the existing ones.

The remainder of this paper is organized as follows. In Sect. 2, one recalls some preliminary results and lemmas for further use. Section 3 analyzes the convergence of the proposed algorithms. In Sect. 4, some numerical examples are presented to illustrate the numerical behavior of the proposed algorithms and compare them with some existing ones. Finally, a brief summary is given in Sect. 5, the last section.

2 Preliminaries

Let C be a convex closed subset of a real Hilbert space $\mathscr{H}$. The weak convergence, which the convergence in the weak topology, and strong convergence (convergence in norm) of $\{x_n\}_{n=1}^{\infty}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y, z \in \mathscr{H}$, we have

(1) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$;
(2) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2, \alpha \in \mathbb{R}$;
(3) $\|\alpha x+\beta y+\gamma z\|^2 = \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2$, where $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha + \beta + \gamma = 1$.

For every point $x \in \mathscr{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$ such that $P_C(x) := \operatorname{argmin}\{\|x - y\|,\ y \in C\}$. P_C is called the metric projection of $\mathscr{H}$ onto C. It is known that P_C is nonexpansive and $P_C(x)$ has the following basic properties:

- $\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \ \forall y \in C$;
- $\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \ \forall y \in \mathscr{H}$.

To prove the convergence of the proposed algorithms, we need the following lemmas.

Lemma 2.1 (Kraikaew and Saejung 2014) *Let $A : \mathscr{H} \to \mathscr{H}$ be a monotone and L-Lipschitz continuous mapping on C. Let $S = P_C(I - \mu A)$, where $\mu > 0$. If $\{x_n\}$ is a sequence in $\mathscr{H}$ satisfying $x_n \rightharpoonup q$ and $x_n - Sx_n \to 0$, then $q \in \mathrm{VI}(C, A) = \mathrm{Fix}(S)$.*

Lemma 2.2 (Maingé 2008) *Assume that $\{a_n\}$ is a nonnegative real number sequence and there is a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1} .$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$.

Lemma 2.3 (Liu 1995; Xu 2002) *Let $\{a_n\}$ be a non-negative real number sequence, which satisfies*

$$a_{n+1} \leq \alpha_n b_n + (1 - \alpha_n)\, a_n, \quad \forall n > 0\,,$$

where $\{\alpha_n\} \subset (0, 1)$ and $\{b_n\}$ are two sequences such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup_{n\to\infty} b_n \leq 0$. Then, $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we introduce two new inertial extragradient algorithms with a new step size for solving variational inequality problems and analyze their convergence. First, we assume that our proposed algorithms satisfy the following conditions.

(C1) The mapping $A : \mathscr{H} \to \mathscr{H}$ is monotone and L-Lipschitz continuous on $\mathscr{H}$.
(C2) The solution set of the (VIP) is nonempty, that is, $\mathrm{VI}(C, A) \neq \emptyset$.
(C3) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ is with the restrictions that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} \alpha_n = 0$. Let $\{\beta_n\} \subset (a, b) \subset (0, 1 - \alpha_n)$ for some $a > 0, b > 0$.

3.1 The Mann-type inertial subgradient extragradient algorithm

Now, we introduce a Mann-type inertial subgradient extragradient algorithm for solving variational inequality problems. The Algorithm 3.1 is read as follows.

Algorithm 3.1 The Mann-type inertial subgradient extragradient algorithm for (VIP)

Initialization: Take $\theta > 0$, $\lambda_1 > 0$, $\mu \in (0, 1)$. Let $x_0, x_1 \in \mathscr{H}$ be arbitrarily fixed.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set

$$w_n = x_n + \theta_n \left(x_n - x_{n-1}\right) ,$$

where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1}\,; \\ \theta, & \text{otherwise}\,. \end{cases} \tag{3.1}$$

Step 2. Compute

$$y_n = P_C\left(w_n - \lambda_n A w_n\right) .$$

If $w_n = y_n$, then stop, and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute

$$z_n = P_{T_n}\left(w_n - \lambda_n A y_n\right) ,$$

where $T_n := \{x \in \mathscr{H} \mid \langle w_n - \lambda_n A w_n - y_n, x - y_n\rangle \leq 0\}$.
Step 4. Compute

$$x_{n+1} = (1 - \alpha_n - \beta_n)\, w_n + \beta_n z_n ,$$

and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\,\|w_n - y_n\|}{\|A w_n - A y_n\|}, \lambda_n\right\}, & \text{if } A w_n - A y_n \neq 0\,; \\ \lambda_n, & \text{otherwise}\,. \end{cases} \tag{3.2}$$

Set $n := n + 1$ and go to **Step 1**.

Remark 3.1 It follows from (3.1) that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| = 0\,.$$

Indeed, we have $\theta_n \|x_n - x_{n-1}\| \leq \epsilon_n$ for all n, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0\,.$$

The following lemmas are quite helpful to analyze the convergence of the algorithm.

Lemma 3.1 *The sequence* $\{\lambda_n\}$ *generated by* (3.2) *is a nonincreasing sequence and*

$$\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}.$$

Proof It follows from (3.2) that $\lambda_{n+1} \leq \lambda_n$ for all $n \in \mathbb{N}$. Hence, $\{\lambda_n\}$ is nonincreasing. On the other hand, we get $\|Aw_n - Ay_n\| \leq L\|w_n - y_n\|$ since A is L-Lipschitz continuous. Consequently,

$$\mu \frac{\|w_n - y_n\|}{\|Aw_n - Ay_n\|} \geq \frac{\mu}{L}, \text{ if } Aw_n \neq Ay_n,$$

which together with (3.2) implies that $\lambda_n \geq \min\{\lambda_1, \frac{\mu}{L}\}$. Since $\{\lambda_n\}$ is nonincreasing and lower bounded, we have $\lim_{n\to\infty} \lambda_n = \lambda \geq \min\left\{\lambda_1, \frac{\mu}{L}\right\}$. □

Lemma 3.2 *Assume that the conditions (C1) and (C2) hold. Let* $\{z_n\}$ *be a sequence generated by Algorithm* 3.1*. Then, for all* $p \in \mathrm{VI}(C, A)$,

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - w_n\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|z_n - y_n\|^2 .$$

Proof By the definition of λ_n, one has

$$\|Aw_n - Ay_n\| \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 0.$$

Using $p \in \mathrm{VI}(C, A) \subset C \subset T_n$, we have

$$\begin{aligned}
2\|z_n - p\|^2 &= 2\left\|P_{T_n}(w_n - \lambda_n Ay_n) - P_{T_n}(p)\right\|^2 \leq 2\langle z_n - p, w_n - \lambda_n Ay_n - p\rangle \\
&= \|z_n - p\|^2 + \|w_n - \lambda_n Ay_n - p\|^2 - \|z_n - w_n + \lambda_n Ay_n\|^2 \\
&= \|z_n - p\|^2 + \|w_n - p\|^2 + \lambda_n^2\|Ay_n\|^2 - 2\langle w_n - p, \lambda_n Ay_n\rangle \\
&\quad - \|z_n - w_n\|^2 - \lambda_n^2\|Ay_n\|^2 - 2\langle z_n - w_n, \lambda_n Ay_n\rangle \\
&= \|z_n - p\|^2 + \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle,
\end{aligned}$$

which implies that

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle. \tag{3.3}$$

We have $\langle Ap, y_n - p\rangle \geq 0$ since $p \in \mathrm{VI}(C, A)$. In addition, since A is monotone, we have $2\lambda_n\langle Ay_n - Ap, y_n - p\rangle \geq 0$. Thus, adding this item to the right side of (3.3), we get

$$\begin{aligned}
\|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \lambda_n Ay_n\rangle + 2\lambda_n\langle Ay_n - Ap, y_n - p\rangle \\
&= \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\langle y_n - z_n, \lambda_n Ay_n\rangle - 2\lambda_n\langle Ap, y_n - p\rangle \\
&\leq \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\lambda_n\langle y_n - z_n, Ay_n - Aw_n\rangle \\
&\quad + 2\lambda_n\langle Aw_n, y_n - z_n\rangle.
\end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned}
2\lambda_n\langle y_n - z_n, Ay_n - Aw_n\rangle &\leq 2\lambda_n\|Ay_n - Aw_n\|\|y_n - z_n\| \leq 2\mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|\|y_n - z_n\| \\
&\leq \mu\frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|^2 + \mu\frac{\lambda_n}{\lambda_{n+1}}\|y_n - z_n\|^2.
\end{aligned} \tag{3.5}$$

Next, we estimate $2\lambda_n \langle Aw_n, y_n - z_n\rangle$. Since $z_n = P_{T_n}(w_n - \lambda_n Ay_n)$ and hence $z_n \in T_n$, we have

$$\langle w_n - \lambda_n Aw_n - y_n, z_n - y_n\rangle \leq 0\,,$$

which implies that

$$\begin{aligned} 2\lambda_n \langle Aw_n, y_n - z_n\rangle &\leq 2\langle y_n - w_n, z_n - y_n\rangle \\ &= \|z_n - w_n\|^2 - \|y_n - w_n\|^2 - \|z_n - y_n\|^2\,. \end{aligned} \tag{3.6}$$

Substituting (3.5) and (3.6) into (3.4), we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|y_n - w_n\|^2 - \left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right)\|z_n - y_n\|^2\,.$$

This completes the proof. □

Theorem 3.1 *Assume that Conditions (C1)–(C3) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $p \in \mathrm{VI}(C, A)$ in norm, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, A)\}$.*

Proof According to Lemma 3.1, it follows that $\lim_{n\to\infty}\left(1 - \mu\frac{\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 0$. Thus, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu\frac{\lambda_n}{\lambda_{n+1}} > 0, \quad \forall n \geq n_0\,. \tag{3.7}$$

Combining Lemma 3.2 and (3.7), we obtain

$$\|z_n - p\| \leq \|w_n - p\|\,, \quad \forall n \geq n_0\,. \tag{3.8}$$

Claim 1. The sequence $\{x_n\}$ is bounded. By the definition of x_{n+1}, one has

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1 - \alpha_n - \beta_n) w_n + \beta_n z_n - p\| \\ &= \|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p) - \alpha_n p\| \\ &\leq \|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\| + \alpha_n\|p\|. \end{aligned} \tag{3.9}$$

On the other hand, it follows from (3.8) that

$$\begin{aligned} &\|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\|^2 \\ &\quad = (1 - \alpha_n - \beta_n)^2\|w_n - p\|^2 + 2(1 - \alpha_n - \beta_n)\beta_n\langle w_n - p, z_n - p\rangle + \beta_n^2\|z_n - p\|^2 \\ &\quad \leq (1 - \alpha_n - \beta_n)^2\|w_n - p\|^2 + 2(1 - \alpha_n - \beta_n)\beta_n\|z_n - p\|\|w_n - p\| + \beta_n^2\|z_n - p\|^2 \\ &\quad \leq (1 - \alpha_n - \beta_n)^2\|w_n - p\|^2 + 2(1 - \alpha_n - \beta_n)\beta_n\|w_n - p\|^2 + \beta_n^2\|w_n - p\|^2 \\ &\quad = (1 - \alpha_n)^2\|w_n - p\|^2\,, \quad \forall n \geq n_0, \end{aligned}$$

which yields

$$\|(1 - \alpha_n - \beta_n)(w_n - p) + \beta_n(z_n - p)\| \leq (1 - \alpha_n)\|w_n - p\|\,,\ \forall n \geq n_0\,. \tag{3.10}$$

Using the definition of w_n, we can write

$$\|w_n - p\| \leq \|x_n - p\| + \alpha_n \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\,. \tag{3.11}$$

By Remark 3.1, we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$. Thus, there exists a constant $M_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq M_1, \quad \forall n \geq 1\,. \tag{3.12}$$

From (3.8), (3.11) and (3.12), we find that

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \alpha_n M_1, \quad \forall n \ge n_0 . \tag{3.13}$$

Combining (3.9), (3.10) and (3.13), we deduce that

$$\begin{aligned}\|x_{n+1} - p\| &\le (1-\alpha_n)\|w_n - p\| + \alpha_n \|p\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n(\|p\| + M_1) \\ &\le \max\{\|x_n - p\|, \|p\| + M_1\} \\ &\le \cdots \le \max\left\{\left\|x_{n_0} - p\right\|, \|p\| + M_1\right\} .\end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded. So the sequences $\{w_n\}$ and $\{z_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&\beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|w_n - y_n\|^2 + \beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n - z_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n(\|p\|^2 + M_2)\end{aligned}$$

for some $M_2 > 0$. Indeed, by the definition of x_{n+1}, one obtains

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\alpha_n-\beta_n)\, w_n + \beta_n z_n - p\|^2 \\ &= \|(1-\alpha_n-\beta_n)(w_n - p) + \beta_n(z_n - p) + \alpha_n(-p)\|^2 \\ &= (1-\alpha_n-\beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 + \alpha_n\|p\|^2 \\ &\quad - \beta_n(1-\alpha_n-\beta_n)\|w_n - z_n\|^2 - \alpha_n(1-\alpha_n-\beta_n)\|w_n\|^2 - \alpha_n\beta_n\|z_n\|^2 \\ &\le (1-\alpha_n-\beta_n)\|w_n - p\|^2 + \beta_n\|z_n - p\|^2 + \alpha_n\|p\|^2 .\end{aligned} \tag{3.14}$$

In view of (3.13), one sees that

$$\begin{aligned}\|w_n - p\|^2 &\le (\|x_n - p\| + \alpha_n M_1)^2 \\ &= \|x_n - p\|^2 + \alpha_n\left(2M_1\|x_n - p\| + \alpha_n M_1^2\right) \\ &\le \|x_n - p\|^2 + \alpha_n M_2\end{aligned} \tag{3.15}$$

for some $M_2 > 0$. Thus, using Lemma 3.2, (3.14) and (3.15), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1-\alpha_n-\beta_n)\|w_n - p\|^2 + \beta_n\|w_n - p\|^2 - \beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|w_n - y_n\|^2 \\ &\quad - \beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n - z_n\|^2 + \alpha_n\|p\|^2 \\ &\le \|x_n - p\|^2 - \beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|w_n - y_n\|^2 \\ &\quad - \beta_n\Big(1-\mu\frac{\lambda_n}{\lambda_{n+1}}\Big)\|y_n - z_n\|^2 + \alpha_n(\|p\|^2 + M_2) .\end{aligned}$$

Claim 3.

$$\begin{aligned}\|x_{n+1} - p\|^2 \le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\Big[&2\beta_n\|w_n - z_n\|\,\|x_{n+1} - p\| \\ &+ 2\langle p, p - x_{n+1}\rangle + \frac{3M\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\Big], \quad \forall n \ge n_0\end{aligned}$$

for some $M > 0$. Indeed, by the definition of w_n, one obtains

$$\begin{aligned}\|w_n - p\|^2 &= \|x_n + \theta_n (x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p\|^2 + 2\theta_n \langle x_n - p, x_n - x_{n-1}\rangle + \theta_n^2 \|x_n - x_{n-1}\|^2 \\ &\leq \|x_n - p\|^2 + 3M\theta_n \|x_n - x_{n-1}\| \,,\end{aligned} \tag{3.16}$$

where $M := \sup_{n\in\mathbb{N}} \{\|x_n - p\|, \theta \|x_n - x_{n-1}\|\} > 0$. Setting $t_n = (1-\beta_n) w_n + \beta_n z_n$, one has

$$\|t_n - w_n\| = \beta_n \|w_n - z_n\| \,. \tag{3.17}$$

It follows from (3.13) that

$$\begin{aligned}\|t_n - p\| &= \|(1-\beta_n)(w_n - p) + \beta_n (z_n - p)\| \\ &\leq (1-\beta_n)\|w_n - p\| + \beta_n \|w_n - p\| \\ &= \|w_n - p\|, \quad \forall n \geq n_0.\end{aligned} \tag{3.18}$$

From (3.16), (3.17) and (3.18), for all $n \geq n_0$, we get

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n) w_n + \beta_n z_n - \alpha_n w_n - p\|^2 \\ &= \|(1-\alpha_n)(t_n - p) - \alpha_n (w_n - t_n) - \alpha_n p\|^2 \\ &\leq (1-\alpha_n)^2 \|t_n - p\|^2 - 2\alpha_n \langle w_n - t_n + p, x_{n+1} - p\rangle \\ &= (1-\alpha_n)^2 \|t_n - p\|^2 + 2\alpha_n \langle w_n - t_n, p - x_{n+1}\rangle + 2\alpha_n \langle p, p - x_{n+1}\rangle \\ &\leq (1-\alpha_n) \|t_n - p\|^2 + 2\alpha_n \|w_n - t_n\| \|x_{n+1} - p\| + 2\alpha_n \langle p, p - x_{n+1}\rangle \\ &\leq (1-\alpha_n) \|x_n - p\|^2 + \alpha_n \Big[2\beta_n \|w_n - z_n\| \|x_{n+1} - p\| \\ &\quad + 2\langle p, p - x_{n+1}\rangle + \frac{3M\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \Big].\end{aligned}$$

Claim 4. The sequence $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.
Case 1. There exists an $N \in \mathbb{N}$, such that $\|x_{n+1} - p\|^2 \leq \|x_n - p\|^2$ for all $n \geq N$. This implies that $\lim_{n\to\infty} \|x_n - p\|^2$ exists. In view of $\lim_{n\to\infty} \left(1 - \mu \frac{\lambda_n}{\lambda_n + 1}\right) = 1 - \mu > 0$ and Condition (C3). It implies from Claim 2 that

$$\lim_{n\to\infty} \|w_n - y_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - z_n\| = 0\,.$$

This implies that $\lim_{n\to\infty} \|z_n - w_n\| = 0$, which together with the boundedness of $\{x_n\}$ gives that

$$\lim_{n\to\infty} \beta_n \|w_n - z_n\| \|x_{n+1} - p\| = 0\,.$$

According to the definition of w_n, one has

$$\|x_n - w_n\| = \theta_n \|x_n - x_{n-1}\| = \alpha_n \cdot \frac{\theta_n}{\alpha_n} \|x_n - x_{n-1}\| \to 0 \quad \text{as } n \to \infty\,.$$

On the other hand, one sees that

$$\|x_{n+1} - w_n\| \leq \alpha_n \|w_n\| + \beta_n \|z_n - w_n\| \to 0 \quad \text{as } n \to \infty\,.$$

This together with $\lim_{n\to\infty} \|x_n - w_n\| = 0$ implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0\,.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, such that $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty} \langle p, p - x_n \rangle = \lim_{j\to\infty} \left\langle p, p - x_{n_j} \right\rangle = \langle p, p - q \rangle .$$

We get $w_{n_j} \rightharpoonup q$ since $\|x_n - w_n\| \to 0$, this together with $\lim_{n\to\infty} \lambda_n = \lambda > 0$ and $\|w_n - y_n\| \to 0$, in the light of Lemma 2.1, yields that $q \in \mathrm{VI}(C, A)$. Since $q \in \mathrm{VI}(C, A)$ and $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, A)\}$, that is $p = P_{\mathrm{VI}(C,A)} 0$, we deduce that

$$\limsup_{n\to\infty} \langle p, p - x_n \rangle = \langle p, p - q \rangle \le 0 .$$

From $\|x_{n+1} - x_n\| \to 0$, we get

$$\limsup_{n\to\infty} \langle p, p - x_{n+1} \rangle \le 0 .$$

Therefore, using Claim 3 and Remark 3.1 in Lemma 2.3, we conclude that $x_n \to p$.
Case 2. There exists a subsequence $\{\|x_{n_j} - p\|^2\}$ of $\{\|x_n - p\|^2\}$ such that $\|x_{n_j} - p\|^2 < \|x_{n_j+1} - p\|^2$ for all $j \in \mathbb{N}$. In this case, it follows from Lemma 2.2 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following inequalities hold for all $k \in \mathbb{N}$:

$$\left\|x_{m_k} - p\right\|^2 \le \|x_{m_k+1} - p\|^2 \text{ and } \|x_k - p\|^2 \le \|x_{m_k+1} - p\|^2 .$$

By Claim 2, we have

$$\begin{aligned}
&\beta_{m_k}\left(1 - \mu \frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right) \left\|w_{m_k} - y_{m_k}\right\|^2 + \beta_{m_k}\left(1 - \mu \frac{\lambda_{m_k}}{\lambda_{m_k+1}}\right) \left\|y_{m_k} - z_{m_k}\right\|^2 \\
&\le \|x_{m_k} - p\|^2 - \|x_{m_k+1} - p\|^2 + \alpha_{m_k}(\|p\|^2 + M_2) \\
&\le \alpha_{m_k}(\|p\|^2 + M_2) .
\end{aligned}$$

Therefore, from Condition (C3), we get

$$\lim_{k\to\infty} \left\|w_{m_k} - y_{m_k}\right\| = 0 \quad \text{and} \quad \lim_{k\to\infty} \left\|y_{m_k} - z_{m_k}\right\| = 0 .$$

As proved in the first case, we get $\|x_{m_k+1} - x_{m_k}\| \to 0$ and $\limsup_{k\to\infty} \langle p, p - x_{m_k+1} \rangle \le 0$. From Claim 3 and $\|x_{m_k} - p\|^2 \le \|x_{m_k+1} - p\|^2$, we obtain

$$\begin{aligned}
\|x_{m_k+1} - p\|^2 \le \left(1 - \alpha_{m_k}\right) \|x_{m_k+1} - p\|^2 + \alpha_{m_k}\Big[&2\beta_{m_k} \left\|w_{m_k} - z_{m_k}\right\| \|x_{m_k+1} - p\| \\
&+ 2\langle p, p - x_{m_k+1} \rangle + \frac{3M\theta_{m_k}}{\alpha_{m_k}} \|x_{m_k} - x_{m_k-1}\|\Big] .
\end{aligned}$$

This implies that

$$\|x_k - p\|^2 \le 2\beta_{m_k} \left\|w_{m_k} - z_{m_k}\right\| \|x_{m_k+1} - p\| + 2\left\langle p, p - x_{m_k+1}\right\rangle + \frac{3M\theta_{m_k}}{\alpha_{m_k}} \|x_{m_k} - x_{m_k-1}\| .$$

Therefore, we obtain $\limsup_{k\to\infty} \|x_k - p\| \le 0$, that is, $x_k \to p$. The proof is completed. □

3.2 The Mann-type inertial Tseng's extragradient algorithm

In this subsection, we introduce a Mann-type inertial Tseng's extragradient algorithm for solving variational inequality problems. Our Algorithm 3.2 is as follows.

The following lemma is very helpful for analyzing the convergence of the Algorithm 3.2.

Algorithm 3.2 The Mann-type inertial Tseng's extragradient algorithm for (VIP)

Initialization: Take $\theta > 0$, $\lambda_1 > 0$, $\mu \in (0,1)$. Let $x_0, x_1 \in \mathscr{H}$ be arbitrary fixed.
Iterative Steps: Calculate x_{n+1} as follows:
Step 1. Given the iterates x_{n-1} and x_n $(n \geq 1)$. Set

$$w_n = x_n + \theta_n \left(x_n - x_{n-1}\right),$$

where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1}; \\ \theta, & \text{otherwise}. \end{cases}$$

Step 2. Compute

$$y_n = P_C\left(w_n - \lambda_n A w_n\right).$$

If $w_n = y_n$, then stop, and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute

$$z_n = y_n - \lambda_n\left(A y_n - A w_n\right),$$

Step 4. Compute

$$x_{n+1} = \left(1 - \alpha_n - \beta_n\right) w_n + \beta_n z_n,$$

and update

$$\lambda_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|A w_n - A y_n\|}, \lambda_n\right\}, & \text{if } A w_n - A y_n \neq 0; \\ \lambda_n, & \text{otherwise}. \end{cases}$$

Set $n := n+1$ and go to **Step 1**.

Lemma 3.3 *Assume that Conditions (C1) and (C2) hold. Let $\{z_n\}$ be a sequence created by Algorithm 3.2. Then,*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right)\|w_n - y_n\|^2, \quad \forall p \in \mathrm{VI}(C, A),$$

and

$$\|z_n - y_n\| \leq \mu \frac{\lambda_n}{\lambda_{n+1}}\|w_n - y_n\|.$$

Proof First, using the definition of λ_n, it is easy to see that

$$\|A w_n - A y_n\| \leq \frac{\mu}{\lambda_{n+1}}\|w_n - y_n\|, \quad \forall n \geq 0. \tag{3.19}$$

By the definition of z_n, one sees that

$$\begin{aligned} \|z_n - p\|^2 &= \|y_n - \lambda_n\left(A y_n - A w_n\right) - p\|^2 \\ &= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle \\ &\quad + \lambda_n^2\|A y_n - A w_n\|^2 - 2\lambda_n\langle y_n - p, A y_n - A w_n\rangle \\ &= \|w_n - p\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle \\ &\quad + \lambda_n^2\|A y_n - A w_n\|^2 - 2\lambda_n\langle y_n - p, A y_n - A w_n\rangle \\ &= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle \\ &\quad + \lambda_n^2\|A y_n - A w_n\|^2 - 2\lambda_n\langle y_n - p, A y_n - A w_n\rangle. \end{aligned} \tag{3.20}$$

Since $y_n = P_C(w_n - \lambda_n A w_n)$, using the property of projection, we obtain

$$\langle y_n - w_n + \lambda_n A w_n, y_n - p\rangle \le 0,$$

or equivalently

$$\langle y_n - w_n, y_n - p\rangle \le -\lambda_n \langle A w_n, y_n - p\rangle . \tag{3.21}$$

From (3.19), (3.20) and (3.21), we have

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\lambda_n \langle A w_n, y_n - p\rangle + \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} \|w_n - y_n\|^2 \\ &\quad - 2\lambda_n \langle y_n - p, A y_n - A w_n\rangle \\ &= \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 - 2\lambda_n \langle y_n - p, A y_n - A p\rangle \\ &\quad - 2\lambda_n \langle y_n - p, A p\rangle .\end{aligned} \tag{3.22}$$

Using $p \in \mathrm{VI}(C, A)$ and the monotonicity of A, we get

$$\langle A p, y_n - p\rangle \ge 0 \ \text{ and } \ \langle A y_n - A p, y_n - p\rangle \ge 0 . \tag{3.23}$$

Combining (3.22) and (3.23), we deduce that

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 .$$

From the definition of z_n and (3.19), we obtain

$$\|z_n - y_n\| \le \mu \frac{\lambda_n}{\lambda_{n+1}} \|w_n - y_n\| .$$

This completes the proof of the lemma. □

Theorem 3.2 *Assume that Conditions (C1)–(C3) hold. Then the sequence $\{x_n\}$ formed by Algorithm 3.2 converges to $p \in \mathrm{VI}(C, A)$ in norm, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, A)\}$.*

Proof Since $\lim_{n\to\infty} \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) = 1 - \mu^2 > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2} > 0, \quad \forall n \ge n_0 . \tag{3.24}$$

Combining Lemma 3.3 and (3.24), we get

$$\|z_n - p\| \le \|w_n - p\| , \quad \forall n \ge n_0 . \tag{3.25}$$

Claim 1. The sequence $\{x_n\}$ is bounded. Using the same arguments with the Claim 1 in the Theorem 3.1, we get that $\{x_n\}$ is bounded. Consequently, the sequences $\{w_n\}$ and $\{z_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&\beta_n \left(1 - \mu^2 \frac{\lambda_n^2}{\lambda_{n+1}^2}\right) \|w_n - y_n\|^2 + \beta_n (1 - \alpha_n - \beta_n) \|w_n - z_n\|^2 \\ &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n (\|p\|^2 + M_2) .\end{aligned}$$

Indeed, by the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|(1-\alpha_n-\beta_n)\,w_n+\beta_n z_n-p\|^2\\ &\le (1-\alpha_n-\beta_n)\,\|w_n-p\|^2+\beta_n\,\|z_n-p\|^2+\alpha_n\|p\|^2\\ &\quad -\beta_n\,(1-\alpha_n-\beta_n)\,\|w_n-z_n\|^2\ .\end{aligned} \tag{3.26}$$

Combining (3.15), Lemma 3.3 and (3.26), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n-\beta_n)\,\|w_n-p\|^2+\beta_n\,\|w_n-p\|^2\\ &\quad -\beta_n\Big(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\,\|w_n-y_n\|^2\\ &\quad +\alpha_n\|p\|^2-\beta_n\,(1-\alpha_n-\beta_n)\,\|w_n-z_n\|^2\\ &\le \|x_n-p\|^2-\beta_n\Big(1-\mu^2\frac{\lambda_n^2}{\lambda_{n+1}^2}\Big)\,\|w_n-y_n\|^2+\alpha_n(\|p\|^2+M_2)\\ &\quad -\beta_n\,(1-\alpha_n-\beta_n)\,\|w_n-z_n\|^2\ .\end{aligned}$$

The desired result can be obtained by a simple deformation.

Claim 3.

$$\begin{aligned}\|x_{n+1}-p\|^2 \le (1-\alpha_n)\,\|x_n-p\|^2+\alpha_n\Big[&2\beta_n\,\|w_n-z_n\|\,\|x_{n+1}-p\|\\ &+2\,\langle p,p-x_{n+1}\rangle+\frac{3M\theta_n}{\alpha_n}\,\|x_n-x_{n-1}\|\Big],\quad \forall n\ge n_0\,.\end{aligned}$$

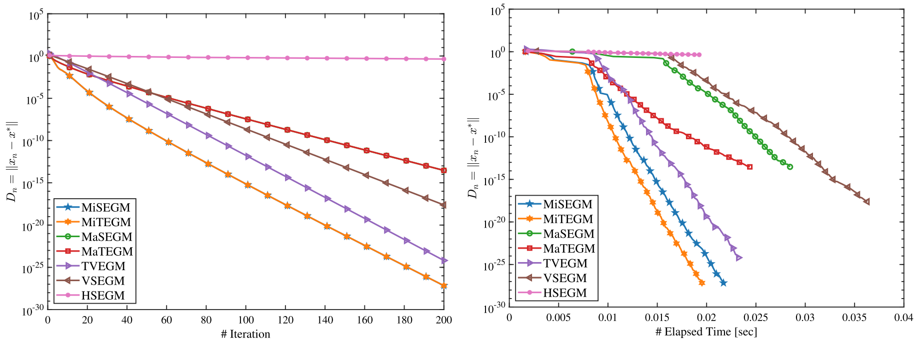
The desired result can be obtained using the same arguments as in the Theorem 3.1 of Claim 3.

Claim 4. The sequence $\{\|x_n-p\|^2\}$ converges to zero, that is, $x_n\to p$. The proof is similar to the Claim 4 of Theorem 3.1, we leave it for the reader to verify. □

4 Numerical examples

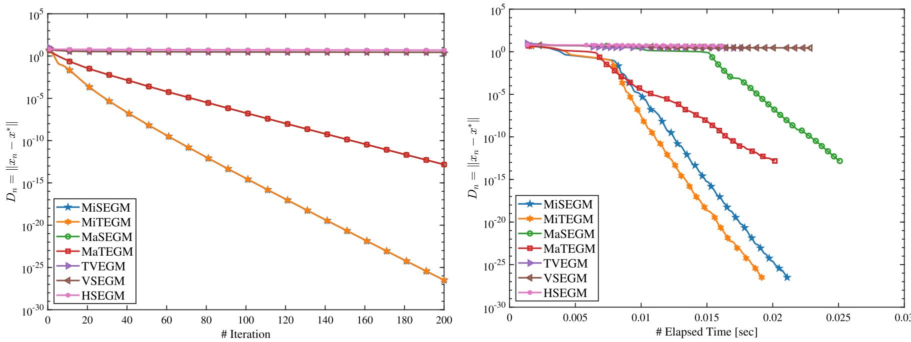
In this section, we provide some numerical examples to show the numerical behavior of our proposed algorithms, namely Algorithm 3.1 (shortly, MiSEGM) and Algorithm 3.2 (MiTEGM), and also to compare them with some existing ones, including the Halpern subgradient extragradient algorithm (HSEGM) (Kraikaew and Saejung 2014), the viscosity subgradient extragradient algorithm (VSEGM) (Shehu and Iyiola 2017), the Tseng's viscosity extragradient algorithm (TVEGM) (Yang and Liu 2019), the Mann-type subgradient extragradient algorithm (MaSEGM) (Thong and Hieu 2019) and the Mann-type Tseng's extragradient algorithm (MaTEGM) (Thong and Hieu 2019). We use the FOM Solver (Beck and Guttmann-Beck 2019) to effectively calculate the projections onto C and T_n. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60 GHz computer with RAM 8.00 GB.

Our parameters are set as follows. In all algorithms, set $\alpha_n = 1/(n+1)$ and $\beta_n = 0.5(1-\alpha_n)$. For the proposed algorithms and the algorithms (MaSEGM) and (MaTEGM), we choose $\lambda_1 = 1$, $\mu = 0.5$. Take $\theta = 0.4$, $\epsilon_n = 100/(n+1)^2$ in our suggested algorithms. For the algorithm (VSEGM), we choose $\ell = 0.5$, $\mu = 0.4$ and $f(x) = 0.9x$. Set $\lambda_0 = 1$, $\mu = 0.5$ and $f(x) = 0.9x$ in the algorithm (TVEGM). For the algorithm (HSEGM), we choose the step size as $\lambda_n = 0.99/L$. Maximum iteration 200 as a common stopping criterion. In our



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 1 Numerical results for Example 4.1 ($x_0 = x_1 = rand(2, 1)$)



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 2 Numerical results for Example 4.1 ($x_0 = x_1 = 5rand(2, 1)$)

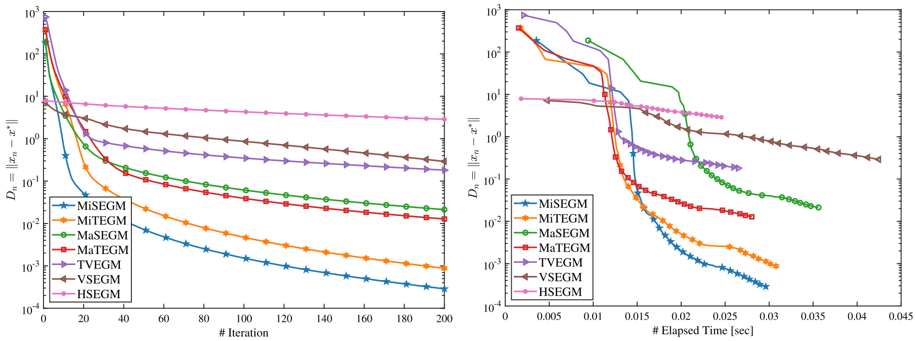
numerical examples, the solution x^* of the problems are known, so we use $D_n = \|x_n - x^*\|$ to measure the n-th iteration error.

Example 4.1 Let us consider the following nonlinear optimization problem via

$$\begin{aligned} \min \quad & F(x) = 1 + x_1^2 - e^{-x_2^2} \\ \text{s.t.} \quad & -5e \le x \le 5e, \end{aligned} \tag{4.1}$$

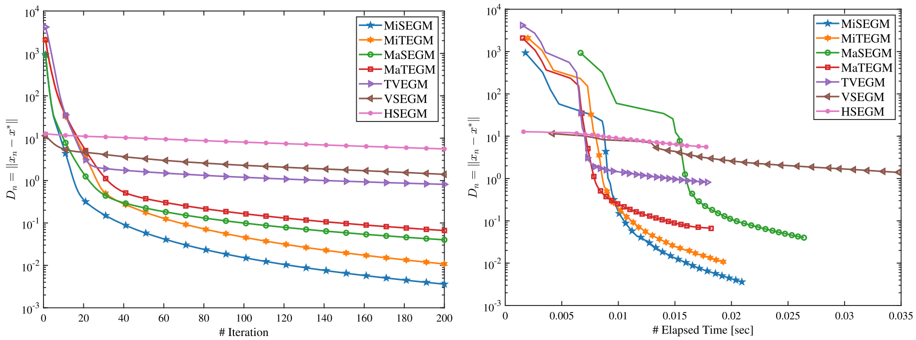
where $x = (x_1, x_2)^{\mathsf{T}} \in R^2$ and $e = (1, 1)^{\mathsf{T}}$. Observe that $\nabla F(x) = (2x_1, 2x_2 e^{-x_2^2})^{\mathsf{T}}$ and the optimal solution for $F(x)$ is $x^* = (0, 0)^{\mathsf{T}}$. Taking $A(x) = \nabla F(x)$, it is easy to check that A is monotone and Lipschitz continuous with constant $L = 2$ on the closed and convex subset $C = \left\{x \in R^2 : -5e \le x \le 5e\right\}$. The initial values $x_0 = x_1$ are randomly generated by $k \times$rand(2,1) in MATLAB. The numerical results are reported in Figs. 1 and 2.

Example 4.2 Consider the linear operator $A : R^m \to R^m$ ($m = 10, 20, 50, 100$) in the form $A(x) = Mx + q$, where $q \in R^m$ and $M = NN^{\mathsf{T}} + U + D$, N is a $m \times m$ matrix, U is a $m \times m$ skew-symmetric matrix, and D is a $m \times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive symmetric definite). The feasible set C is given by $C = \{x \in R^m : -2 \le x_i \le 5,\ i = 1, \dots, m\}$. It is clear that A is monotone and Lipschitz continuous with constant $L = \|M\|$. In this experiment, all entries of N, D are generated



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 3 Numerical results for Example 4.2 ($m = 10$)



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 4 Numerical results for Example 4.2 ($m = 20$)

randomly in $[0, 2]$ and U is generated randomly in $[-2, 2]$. Let $q = 0$, then the solution set is $x^* = \{\mathbf{0}\}$. The initial values $x_0 = x_1$ are randomly generated by *5rand(m,1)* in MATLAB. The numerical results of all the algorithms in different dimensions are shown in Figs. 3, 4, 5 and 6.

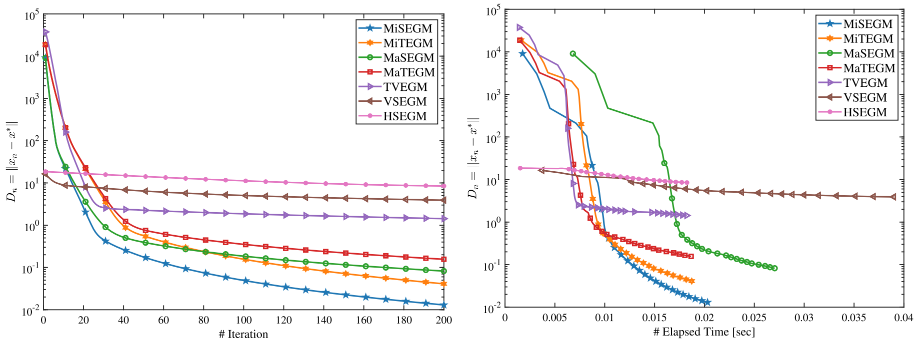
Example 4.3 Finally, we consider our problem in the Hilbert space $\mathscr{H} = L^2([0, 1])$ with the inner product $\langle x, y\rangle := \int_0^1 x(t)y(t)\,\mathrm{d}t$ and the induced norm $\|x\| := (\int_0^1 |x(t)|^2\,\mathrm{d}t)^{1/2}$, $\forall x, y \in \mathscr{H}$. Let the feasible set be the unit ball $C := \{x \in \mathscr{H} : \|x\| \leq 1\}$. Define an operator $A : C \to \mathscr{H}$ by

$$(Ax)(t) = \int_0^1 (x(t) - G(t, s)g(x(s)))\,\mathrm{d}s + h(t), \quad t \in [0, 1],\ x \in C\,,$$

where

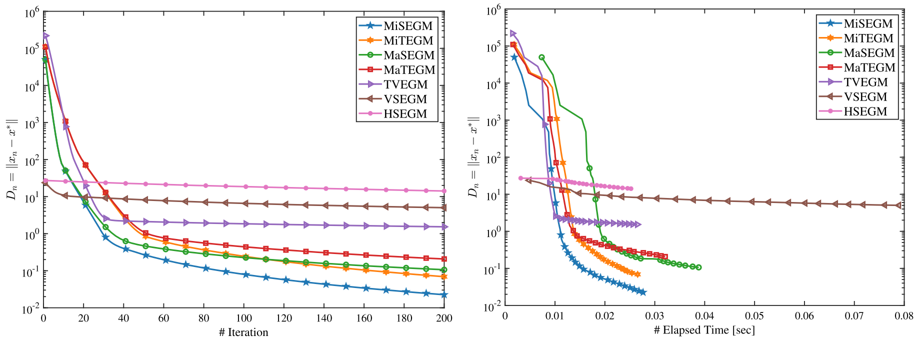
$$G(t, s) = \frac{2ts\mathrm{e}^{t+s}}{e\sqrt{\mathrm{e}^2 - 1}}\,, \quad g(x) = \cos x\,, \quad h(t) = \frac{2t\mathrm{e}^t}{e\sqrt{\mathrm{e}^2 - 1}}\,.$$

It is known that A is monotone and L-Lipschitz continuous with $L = 2$ and $x^*(t) = \{\mathbf{0}\}$ is the solution of the corresponding variational inequality problem. Note that the projection on



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 5 Numerical results for Example 4.2 ($m = 50$)



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 6 Numerical results for Example 4.2 ($m = 100$)

C is inherently explicit, that is,

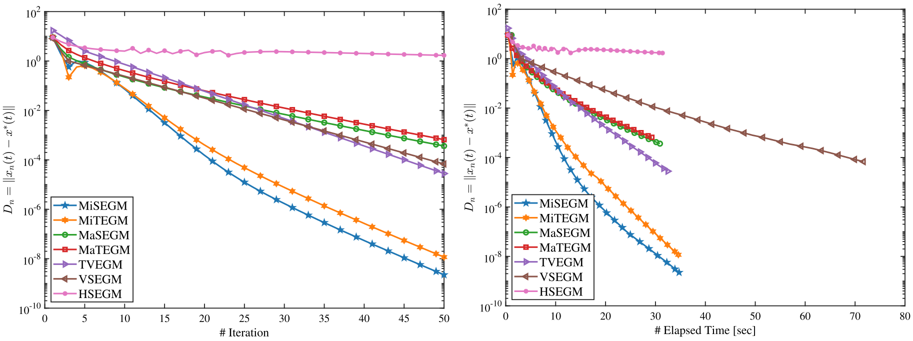
$$P_C(x) = \begin{cases} \dfrac{x}{\|x\|_{L^2}}, & \text{if } \|x\|_{L^2} > 1\,; \\ x, & \text{if } \|x\|_{L^2} \leq 1\,. \end{cases}$$

We choose the maximum iteration of 50 as a common stopping criterion. Figs. 7, 8, 9 and 10 show the numerical behaviors of all the algorithms with four starting points $x_0(t) = x_1(t)$.

Remark 4.1 (1) From Figs. 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10, we know that the proposed algorithms outperformance the existing algorithms in terms of the number of iteration and the elapsed time. In addition, these observed results have nothing to do with the size of the dimension and the selection of initial values.

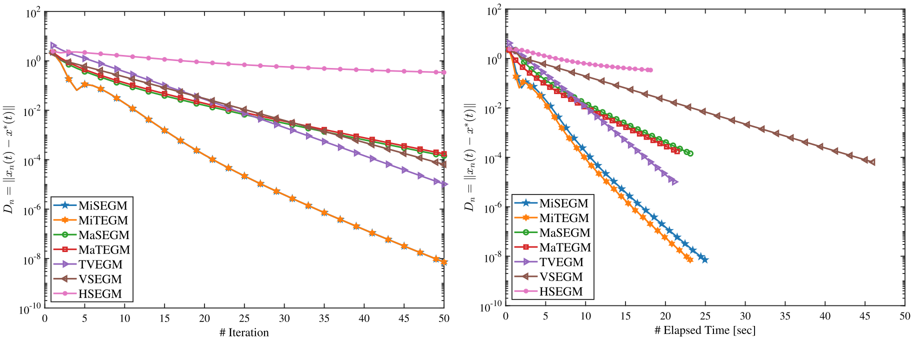
(2) The maximum number of iterations we chose was only 200. Note that the iteration error of algorithm (HSEGM) is very big. In actual applications, it may require more iterations to meet the accuracy requirements.

(3) It should be pointed out that since the algorithm (VSEGM) uses the Armijo-like step size rule, which leads to taking more execution time.



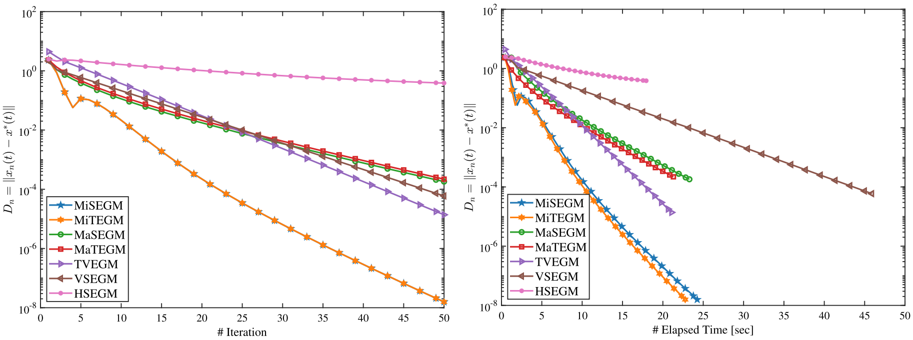
(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 7 Numerical results for Example 4.3 ($x_0(t) = x_1(t) = 10e^t$)



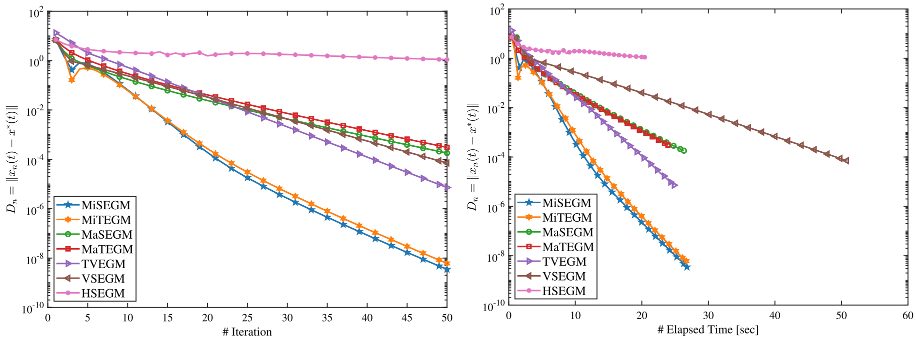
(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 8 Numerical results for Example 4.3 ($x_0(t) = x_1(t) = 5\cos(t)$)



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 9 Numerical results for Example 4.3 ($x_0(t) = x_1(t) = 10t^2$)



(a) Comparison of the number of iterations **(b)** Comparison of the elapsed time

Fig. 10 Numerical results for Example 4.3 ($x_0(t) = x_1(t) = 10 \log(t)$)

5 Conclusions

In this paper, we presented two new inertial extragradient algorithms with a new step size for finding the solution set of a monotone, Lipschitz-continuous variational inequality problems in real Hilbert spaces. We proved strong convergence theorems of the proposed algorithms under some mild conditions imposed on parameters. Some numerical examples of finite and infinite dimensions were performed to illustrate the performance of the suggested algorithms and compare them with previously known ones. The two algorithms obtained in this paper improved and extended the results of some existing literature.

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