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ORIGINAL PAPER

Two projection-based methods for bilevel pseudomonotone variational inequalities involving non-Lipschitz operators

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Abstract
In this paper, we propose two new iterative algorithms to discover solutions of bilevel pseudomonotone variational inequalities with non-Lipschitz continuous operators in real Hilbert spaces. Our proposed algorithms need to compute the projection on the feasible set only once in each iteration although they employ Armijo line search methods. Strong convergence theorems of the suggested algorithms are established under suitable and weaker conditions. Some numerical experiments and applications are given to demonstrate the performance of the offered algorithms compared to some known ones.

Keywords Bilevel variational inequality · Inertial method · Armijo stepsize · Pseudomonotone mapping · Non-Lipschitz operator

Mathematics Subject Classification 47J20 · 47J25 · 47J30 · 68W10 · 65K15

1 Introduction

Bilevel optimization problems are hierarchical optimization problems in which the feasible region of the upper-level problem is restricted by the solution set of the lower-level problem. For more details on the theory, algorithms and applications of bilevel optimization problems, we refer the reader to the recent monograph [1]. In this paper, we focus on a special case of the bilevel optimization problem, namely the bilevel variational inequality problem (shortly, BVIP), where both the upper- and lower-level problems are restricted by variational inequalities. BVIPs cover a number of nonlinear optimization problems, such as fixed point problems, quasi-variational inequality problems, complementary problems, saddle problems, and minimum norm problems. Let C be a nonempty, closed, and convex subset of a real Hilbert

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space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Recall that the BVIP is described as follows:

$$\text{find } x^* \in \Omega \text{ such that } \left\langle Fx^*, y - x^* \right\rangle \geq 0, \quad \forall y \in \Omega, \tag{BVIP}$$

where $F : C \to \mathcal{H}$ is an operator and Ω denotes the set of all solutions of the following variational inequality problem (shortly, VIP):

$$\text{find } y^* \in C \text{ such that } \left\langle My^*, z - y^* \right\rangle \geq 0, \quad \forall z \in C, \tag{VIP}$$

where $M : C \to \mathcal{H}$ is an operator. It is known that VIPs play a significant role in applied science and optimization theory. They provide a general and useful framework for solving engineering problems, data sciences, and other fields; see, e.g., [2–4]. Thus, numerical methods for studying variational inequalities have attracted numerous interest among researchers.

In this paper, we are concerned with projection-based methods for solving the variational inequality problem (VIP). The simplest projection-type method is the projected gradient method (shortly, PGM). However, the weak convergence of the PGM requires that the operator M involved is Lipschitz continuous and strongly monotone. Korpelevich [5] proposed a method called the extragradient method (EGM) to overcome the drawbacks of the PGM. It is known that the EGM converges weakly to the solution of (VIP) under the condition that the mapping M is monotone and Lipschitz continuous. In recent years, the EGM was extensively studied by scholars, and they proposed a large number of improved versions of the EGM for solving variational inequalities in infinite-dimensional Hilbert spaces; see, e.g., [6–9] and the references therein. On the other hand, the EGM and some of its improved methods will fail if operator M does not satisfy Lipschitz continuity. To overcome this difficulty, Iusem [10] proposed a new iterative algorithm that is based on the EGM and the Armijo line search method for solving variational inequality problems in finite-dimensional spaces. Note that the convergence of Iusem's method is proved under the assumption that the mapping M is not Lipschitz continuous. It should be noted that Iusem's method may need to compute multiple projections on the feasible set in each iteration due to its use of the Armijo line search criterion. Solodov and Svaiter [11] introduced an improved algorithm with a new Armijo-type step size to overcome this obstacle. They construct a new hyperplane which separates the current iterate from the solution of (VIP). The convergence of the method is also confirmed under the condition that the mapping M is uniformly continuous. Moreover, the method of Solodov and Svaiter [11] requires only one projection onto the feasible set in each iteration, which greatly improves the computational efficiency of the method of Iusem [10]. Recently, a large number of improved algorithms of Solodov and Svaiter [11] were proposed to solve monotone VIPs (see, e.g., [12–14]) and pseudomonotone VIPs (see, e.g., [15–19]). The convergence of these methods is established under the assumption of the mapping M without Lipschitz continuity.

Next we state some algorithms for solving the bilevel variational inequality problem (BVIP), and these motivate us to develop several new efficient iterative schemes. Yamada [20] investigated a bilevel problem associated with the BVIP, which is described as follows:

$$\text{find } x^* \in \operatorname{Fix}(T) \text{ such that } \langle Fx^*, \ y - x^* \rangle \geq 0, \quad \forall y \in \operatorname{Fix}(T), \tag{BVIFPP}$$

where $\operatorname{Fix}(T) := \{x \in \mathcal{H} : Tx = x\}$ denotes the set of fixed points of the nonexpansive mapping T, and the mapping F is Lipschitz continuous and inverse strongly monotone. Let $\lambda > 0$. If we set $Tx = P_C(x - \lambda Mx)$, then $x \in \operatorname{Fix}(T) \Leftrightarrow x \in \Omega$ and thus the problem (BVIP) becomes the problem (BVIFPP). Yamada [20] introduced the hybrid steepest descent method $x_{n+1} = (I - \alpha_{n+1}\lambda F)(Tx_n)$ for solving (BVIFPP), where $\{\lambda\}$ and $\{\alpha_{n+1}\}$ are suitable sequences that satisfy some conditions. He proved that the iterative sequence generated by the method converges strongly to the unique solution of (BVIFPP). Recently, a number of numerical algorithms that based on the hybrid steepest descent method were presented for solving the monotone (BVIP) (see, e.g., [21]) and the pseudomonotone (BVIP) (see, e.g., [22–24]). A common characteristic enjoyed by these algorithms is that the operator M is required to be Lipschitz continuous. In recent years, the study of acceleration algorithms has attracted a great interest among researchers which is due to the need of practical problems. Recently, scholars proposed a large number of acceleration algorithms based on inertial techniques (see [25,26] for more detials) to address variational inequalities, splitting problems, fixed point problems, and a variety of optimization problems; see, e.g., [27–32] and the references therein. A common feature of these inertial algorithms is that the next iteration depends on the combination of the previous two iterations. This small change greatly improves the computational efficiency of inertial type algorithms.

Motivated and inspired by the works in [17,18,20] and by the ongoing research in these directions, in this paper, we propose two accelerated projection-based methods for solving the bilevel variational inequality problem (BVIP) with a pseudomonotone and uniformly continuous operator. The paper is organized as follows. In the next section, we review some definitions and lemmas that need to be used in the sequel. Section 3 states the suggested iterative schemes and analyzes their convergence properties. In Sect. 4, we perform some numerical examples to demonstrate the advantages of the proposed algorithms in comparison with some related ones. Finally, we conclude the paper with a brief summary in Sect. 5, the last section.

2 Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. Let $P_C : \mathcal{H} \to C$ denote the metric (nearest point) projection from $\mathcal{H}$ onto C, characterized by $P_C(x) := \arg\min\{\|x - y\|,\ y \in C\}$. It is known that P_C is nonexpansive and $P_C(x) \in C$ for all $x \in \mathcal{H}$. Recall that a mapping $M : \mathcal{H} \to \mathcal{H}$ is said to be:

(i) *L-Lipschitz continuous* with $L > 0$ if $\|Mx - My\| \leq L\|x - y\|,\ \forall x, y \in \mathcal{H}$ (if $L \in (0, 1)$ then mapping M is called a *contraction*. In particular, when $L = 1$, mapping M is said to be *nonexpansive*).

(ii) *α-strongly monotone* if there exists a constant $\alpha > 0$ such that $\langle Mx - My, x - y\rangle \geq \alpha\|x - y\|^2,\ \forall x, y \in \mathcal{H}$.

(iii) *monotone* if $\langle Mx - My, x - y\rangle \geq 0,\ \forall x, y \in \mathcal{H}$.

(iv) *pseudomonotone* if $\langle Mx, y - x\rangle \geq 0 \Rightarrow \langle My, y - x\rangle \geq 0,\ \forall x, y \in \mathcal{H}$.

(v) *sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies $\{Mx_n\}$ converges weakly to Mx.

The following lemmas will be used in the convergence analysis of our algorithms.

Lemma 2.1 *([33])* Assume that C is a closed and convex subset of a real Hilbert space $\mathcal{H}$. Let operator $M : C \to \mathcal{H}$ be continuous and pseudomonotone. Then, x^* is a solution of (VIP) if and only if $\langle Mx, x - x^*\rangle \geq 0,\ \forall x \in C$.

Lemma 2.2 *([34])* Assume that C is a convex and closed nonempty subset of a real Hilbert space $\mathcal{H}$. Let h be a real-valued function on $\mathcal{H}$ and define $K = \{x \in C : h(x) \leq 0\}$. If K is nonempty and h is θ-Lipschitz continuous on C, then

$$\operatorname{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\}, \quad \forall x \in C,$$

where $\operatorname{dist}(x, K)$ denotes the distance function from x to K.

Lemma 2.3 ([20]) Let $\gamma > 0$ and $\alpha \in (0, 1]$. Let $F : \mathcal{H} \to \mathcal{H}$ be a β-strongly monotone and L-Lipschitz continuous mapping with $0 < \beta \leq L$. Associating with a nonexpansive mapping $T : \mathcal{H} \to \mathcal{H}$, define a mapping $T^\gamma : \mathcal{H} \to \mathcal{H}$ by $T^\gamma x = (I - \alpha\gamma F)(Tx), \forall x \in \mathcal{H}$. Then, T^γ is a contraction provided $\gamma < \frac{2\beta}{L^2}$, that is,

$$\|T^\gamma x - T^\gamma y\| \leq (1 - \alpha\eta)\|x - y\|, \quad \forall x, y \in \mathcal{H},$$

where $\eta = 1 - \sqrt{1 - \gamma\left(2\beta - \gamma L^2\right)} \in (0, 1)$.

Lemma 2.4 ([35]) Let $\{p_n\}$ be a positive sequence, $\{q_n\}$ be a sequence of real numbers, and $\{\alpha_n\}$ be a sequence in $(0, 1)$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$. Assume that

$$p_{n+1} \leq (1 - \alpha_n)p_n + \alpha_n q_n, \quad \forall n \geq 1.$$

If $\limsup_{k\to\infty} q_{n_k} \leq 0$ for every subsequence $\left\{p_{n_k}\right\}$ of $\{p_n\}$ satisfying $\liminf_{k\to\infty} (p_{n_k+1} - p_{n_k}) \geq 0$, then $\lim_{n\to\infty} p_n = 0$.

3 Main results

In this section, we introduce two new algorithms for finding the solutions of the bilevel pseudomonotone variational inequality problem (BVIP). The following assumptions are assumed to be satisfied before introducing our algorithms.

(A1) The feasible set C is a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$.
(A2) The solution set of the problem (VIP) is nonempty, that is, $\Omega \neq \emptyset$.
(A3) The operator $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, uniformly continuous on $\mathcal{H}$, and the operator $M : \mathcal{H} \to \mathcal{H}$ satisfies the following assumption

$$\text{whenever } \{x_n\} \subset C,\ x_n \rightharpoonup z,\ \text{one has } \|Mz\| \leq \liminf_{n\to\infty} \|Mx_n\|. \tag{3.1}$$

(A4) The mapping $F : \mathcal{H} \to \mathcal{H}$ is L_F-Lipschitz continuous and β-strongly monotone on $\mathcal{H}$ such that $L_F \geq \beta$.
(A5) Let $\{\epsilon_n\}$ be a positive sequence such that $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Remark 3.1 Note that the assumption (3.1) is used by many recent works on pseudomonotone variational inequalities (see, e.g., [18,36]). It is easy to check that Assumption (3.1) is weaker than the sequential weak continuity of the mapping M (see [36, Remark 3.2]). Moreover, it is not necessary to impose Assumption (3.1) when mapping M is monotone (see [9,37]). On the other hand, the solution of problem (BVIP) is unique provided that Conditions (A1)–(A4) are satisfied (see [21,38] for more details).

3.1 First type of projection algorithm

Based on the inertial method, the Algorithm 3.3 of Thong et al. [17] and the hybrid steepest descent method [20], we introduce a new iterative scheme containing only one projection on the feasible set to address the bilevel variational inequality problem (BVIP) with a pseudomonotone and uniformly continuous operator. We now state the scheme in Algorithm 3.1.

Algorithm 3.1 Inertial extragradient method for solving (BVIP).

Initialization: Take $\theta > 0$, $\ell \in (0, 1)$, $\mu > 0$, $\lambda \in (0, 1/\mu)$, $\gamma \in (0, 2\beta/L_F^2)$ and let $x_0, x_1 \in \mathcal{H}$.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$, where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1}; \\ \theta, & \text{otherwise.} \end{cases} \tag{In-Cri}$$

Step 2. Compute $y_n = P_C(w_n - \lambda M w_n)$. Set $r_\lambda(w_n) = w_n - y_n$.
Step 3. Compute $t_n = w_n - \tau_n r_\lambda(w_n)$, where $\tau_n = \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\left\langle M w_n - M\left(w_n - \ell^m r_\lambda(w_n)\right), r_\lambda(w_n)\right\rangle \leq \mu \|r_\lambda(w_n)\|^2. \tag{Ar-1}$$

Step 4. Compute $z_n = P_{H_n}(w_n)$, where the half-space H_n is defined by

$$H_n = \{x \in C : h_n(x) \leq 0\} \quad \text{and} \quad h_n(x) = \langle M t_n, x - t_n\rangle. \tag{Hn-1}$$

Step 5. Compute $x_{n+1} = z_n - \alpha_n \gamma F z_n$.
Set $n := n + 1$ and go to **Step 1**.

Remark 3.2 We note here that the inertial calculation criterion (In-Cri) is easy to implement since the term $\|x_n - x_{n-1}\|$ is known before calculating θ_n. It follows from (In-Cri) and the assumptions on $\{\alpha_n\}$ that $\lim_{n\to\infty}(\theta_n\|x_n - x_{n-1}\|)/\alpha_n = 0$. Furthermore, the assumption (A5) is easily satisfied by, for example, taking $\alpha_n = 1/(n + 1)$ and $\epsilon_n = 1/(n + 1)^2$.

The following lemmas are crucial for the convergence analysis of Algorithm 3.1.

Lemma 3.1 *Suppose that Assumptions (A1)–(A3) hold. The Armijo line search rule (Ar-1) is well defined.*

Proof Since mapping M is uniformly continuous on C and $\ell \in (0, 1)$, one obtains

$$\lim_{m\to\infty}\left\langle M w_n - M\left(w_n - \ell^m r_\lambda(w_n)\right), r_\lambda(w_n)\right\rangle = 0.$$

Moreover, it can be easily seen that $\|r_\lambda(w_n)\| > 0$ (otherwise, y_n is a solution of (VIP)). Thus, there exists a non-negative integer m_n satisfying (Ar-1). □

Lemma 3.2 *Suppose that Assumptions (A1)–(A3) hold. Let x^* be a solution of (VIP). Then $h_n(x^*) \leq 0$ and $h_n(w_n) \geq \tau_n\left(\lambda^{-1} - \mu\right)\|r_\lambda(w_n)\|^2$. In particular, if $r_\lambda(w_n) \neq 0$ then $h_n(w_n) > 0$.*

Proof From $x^* \in \Omega$, $t_n \in C$ and Lemma 2.1, one obtains $h_n(x^*) = \langle M t_n, x^* - t_n\rangle \leq 0$. Using the definitions of h_n and t_n, one sees that

$$h_n(w_n) = \langle M t_n, w_n - t_n\rangle = \langle M t_n, \tau_n r_\lambda(w_n)\rangle = \tau_n\langle M t_n, r_\lambda(w_n)\rangle. \tag{3.2}$$

By using the property of projection $\|x-P_C(y)\|^2 \leq \langle x-y, x-P_C(y)\rangle, \forall x \in C, y \in \mathcal{H}$ and taking $x=w_n$ and $y=w_n-\lambda M w_n$, we obtain

$$\|w_n-P_C(w_n-\lambda M w_n)\|^2 \leq \lambda \langle M w_n, w_n-P_C(w_n-\lambda M w_n)\rangle,$$

which yields that $\langle M w_n, r_\lambda(w_n)\rangle \geq \lambda^{-1}\|r_\lambda(w_n)\|^2$. From (Ar-1), one has

$$\begin{aligned}\langle M t_n, r_\lambda(w_n)\rangle &\geq \langle M w_n, r_\lambda(w_n)\rangle-\mu\|r_\lambda(w_n)\|^2\\ &\geq \left(\lambda^{-1}-\mu\right)\|r_\lambda(w_n)\|^2.\end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we observe that $h_n(w_n) \geq \tau_n\left(\lambda^{-1}-\mu\right)\|r_\lambda(w_n)\|^2$. □

Lemma 3.3 *Suppose that Assumptions (A1)–(A3) hold. Let $\{w_n\}$ and $\{y_n\}$ be two sequences formulated by Algorithm 3.1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $\{w_{n_k}\}$ converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0$, then $z \in \Omega$.*

Proof The proof of this lemma follows that of Lemma 3.11 in [17], and so it is omitted. □

Lemma 3.4 *Suppose that Assumptions (A1)–(A3) hold. Let the sequences $\{w_n\}$ and $\{y_n\}$ be created by Algorithm 3.1. If $\lim_{n\to\infty}\tau_n\|r_\lambda(w_n)\|^2=0$ then $\lim_{n\to\infty}\|w_n-y_n\|=0$.*

Proof We show that $\lim_{n\to\infty}\|w_n-y_n\|=0$ by consider two cases of τ_n. First, we assume that $\liminf_{n\to\infty}\tau_n>0$. Then, there exists a positive number τ such that $\tau_n \geq \tau>0, \forall k \in \mathbb{N}$. Moreover, one sees that

$$\|w_n-y_n\|^2=\frac{1}{\tau_n}\tau_n\|w_n-y_n\|^2 \leq \frac{1}{\tau}\cdot\tau_n\|w_n-y_n\|^2.$$

Therefore, we obtain $\lim_{n\to\infty}\|w_n-y_n\|=0$ by the hypothesis. On the other hand, one supposes that $\liminf_{n\to\infty}\tau_n=0$. In this situation, we suppose that $\{n_k\}$ is a subsequence of $\{n\}$ such that

$$\lim_{k\to\infty}\tau_{n_k}=0 \text{ and } \lim_{k\to\infty}\left\|w_{n_k}-y_{n_k}\right\|=a>0. \tag{3.4}$$

Let $y_k=w_{n_k}-\ell^{-1}\tau_{n_k}(w_{n_k}-y_{n_k})$. It follows that

$$\lim_{k\to\infty}\left\|y_k-w_{n_k}\right\|^2=\lim_{k\to\infty}\frac{1}{\ell^2}\tau_{n_k}\cdot\tau_{n_k}\left\|w_{n_k}-y_{n_k}\right\|^2=0,$$

which together with the fact that M is uniformly continuous, gives $\lim_{k\to\infty}\left\|M w_{n_k}-M y_k\right\| = 0$. From the definition of y_k and (Ar-1), we obtain

$$\left\langle M w_{n_k}-M y_k, w_{n_k}-y_{n_k}\right\rangle>\mu\left\|w_{n_k}-y_{n_k}\right\|^2, \tag{3.5}$$

which further yields that $\lim_{k\to\infty}\left\|w_{n_k}-y_{n_k}\right\|=0$. This contradicts the hypothesis (3.4). Thus we conclude that $\lim_{n\to\infty}\|w_n-y_n\|=0$. The proof is completed. □

Theorem 3.1 *Assume that Assumptions (A1)–(A5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to the unique solution of the (BVIP) in norm.*

Proof We divide the proof into four claims.

Claim 1. The sequence $\{x_n\}$ is bounded. Let $p \in \Omega$. From Lemma 2.2 and the property of projection $\|P_C(x)-y\|^2 \leq \|x-y\|^2-\|x-P_C(x)\|^2, \forall y \in C$, and take $x=w_n, y=p$ and $C=H_n$, we deduce

$$\begin{aligned}\|z_n-p\|^2 &\leq \|w_n-p\|^2-\left\|w_n-P_{H_n}(w_n)\right\|^2\\ &=\|w_n-p\|^2-\operatorname{dist}^2(w_n, H_n),\end{aligned} \tag{3.6}$$

which means that

$$\|z_n - p\| \le \|w_n - p\|, \quad \forall n \ge 1. \tag{3.7}$$

By the definition of w_n, one has

$$\|w_n - p\| \le \alpha_n \cdot \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| + \|x_n - p\|. \tag{3.8}$$

According to Remark 3.2, we have $\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$ as $n \to \infty$. Therefore, there exists a constant $Q_1 > 0$ such that

$$\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \le Q_1, \quad \forall n \ge 1. \tag{3.9}$$

Combining (3.7), (3.8) and (3.9), we obtain

$$\|z_n - p\| \le \|w_n - p\| \le \|x_n - p\| + \alpha_n Q_1, \quad \forall n \ge 1. \tag{3.10}$$

Using Lemma 2.3 and (3.10), it follows that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(I - \alpha_n \gamma F) z_n - (I - \alpha_n \gamma F) p - \alpha_n \gamma F p\| \\
&\le (1 - \alpha_n \eta)\|z_n - p\| + \alpha_n \gamma \|Fp\| \\
&\le (1 - \alpha_n \eta)\|x_n - p\| + \alpha_n \eta \cdot \frac{Q_1}{\eta} + \alpha_n \eta \cdot \frac{\gamma}{\eta}\|Fp\| \\
&\le \max\left\{\frac{Q_1 + \gamma\|Fp\|}{\eta}, \|x_n - p\|\right\} \\
&\le \cdots \le \max\left\{\frac{Q_1 + \gamma\|Fp\|}{\eta}, \|x_1 - p\|\right\},
\end{aligned}$$

where $\eta = 1 - \sqrt{1 - \gamma\left(2\beta - \gamma L_F^2\right)} \in (0, 1)$. This implies that the sequence $\{x_n\}$ is bounded. We obtain that the sequences $\{w_n\}$, $\{y_n\}$, $\{t_n\}$, and $\{z_n\}$ are also bounded.
Claim 2.

$$\|z_n - w_n\|^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n Q_4$$

and

$$\left[D^{-1}\tau_n\left(\lambda^{-1} - \mu\right)\|r_\lambda(w_n)\|^2\right]^2 \le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n Q_4$$

for some $Q_4 > 0$. It follows from (3.10) that

$$\begin{aligned}
\|w_n - p\|^2 &\le \|x_n - p\|^2 + \alpha_n\left(2Q_1\|x_n - p\| + \alpha_n Q_1^2\right) \\
&\le \|x_n - p\|^2 + \alpha_n Q_3
\end{aligned} \tag{3.11}$$

for some $Q_3 > 0$. Using the inequality $\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle$, $\forall x, y \in \mathcal{H}$, one has

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|(I - \alpha_n \gamma F) z_n - (I - \alpha_n \gamma F) p - \alpha_n \gamma F p\|^2 \\
&\le (1 - \alpha_n \eta)^2\|z_n - p\|^2 + 2\alpha_n \gamma \langle Fp, p - x_{n+1}\rangle \\
&\le \|z_n - p\|^2 + \alpha_n Q_2
\end{aligned} \tag{3.12}$$

for some $Q_2 > 0$. Combining (3.6), (3.11) and (3.12), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|w_n-p\|^2-\|z_n-w_n\|^2+\alpha_n Q_2 \\ &\leq \|x_n-p\|^2-\|z_n-w_n\|^2+\alpha_n Q_4,\end{aligned} \tag{3.13}$$

where $Q_4 := Q_2 + Q_3$. The first inequality can be obtained by a simple conversion.

From $\{Mt_n\}$ is bounded, there is $D > 0$ such that $\|Mt_n\| \leq D, \forall n$. For any $u, v \in \mathcal{H}$, we derive

$$\|h_n(u)-h_n(v)\| = \|\langle Mt_n, u-v\rangle\| \leq \|Mt_n\|\,\|u-v\| \leq D\|u-v\|,$$

which means that $h_n(x)$ is D-Lipschitz continuous on $\mathcal{H}$. From Lemmas 2.2 and 3.2, we find that

$$\operatorname{dist}(w_n, H_n) \geq D^{-1}h_n(w_n) \geq D^{-1}\tau_n\left(\lambda^{-1}-\mu\right)\|r_\lambda(w_n)\|^2.$$

This together with (3.6) gives

$$\|z_n-p\|^2 \leq \|w_n-p\|^2-\left[D^{-1}\tau_n\left(\lambda^{-1}-\mu\right)\|r_\lambda(w_n)\|^2\right]^2.$$

From (3.11) and (3.12), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq \|z_n-p\|^2+\alpha_n Q_2 \\ &\leq \|x_n-p\|^2-\left[D^{-1}\tau_n\left(\lambda^{-1}-\mu\right)\|r_\lambda(w_n)\|^2\right]^2+\alpha_n Q_4.\end{aligned}$$

A simple transformation of the above equation can obtain the second inequality.

Claim 3.

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n\eta)\|x_n-p\|^2+\alpha_n\eta\left[\frac{2\gamma}{\eta}\langle Fp, p-x_{n+1}\rangle+\frac{3Q\theta_n}{\alpha_n\eta}\|x_n-x_{n-1}\|\right]$$

for some $Q > 0$. Indeed, we have

$$\|w_n-p\|^2 \leq \|x_n-p\|^2+2\theta_n\|x_n-p\|\|x_n-x_{n-1}\|+\theta_n^2\|x_n-x_{n-1}\|^2. \tag{3.14}$$

Combining (3.7) and (3.12), we deduce

$$\|x_{n+1}-p\|^2 \leq (1-\alpha_n\eta)\|w_n-p\|^2+2\alpha_n\gamma\langle Fp, p-x_{n+1}\rangle. \tag{3.15}$$

Substituting (3.14) into (3.15), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\leq (1-\alpha_n\eta)\|x_n-p\|^2+2\alpha_n\gamma\langle Fp, p-x_{n+1}\rangle \\ &\quad+\theta_n\|x_n-x_{n-1}\|\left(2\|x_n-p\|+\theta\|x_n-x_{n-1}\|\right) \\ &\leq (1-\alpha_n\eta)\|x_n-p\|^2+\alpha_n\eta\left[\frac{2\gamma}{\eta}\langle Fp, p-x_{n+1}\rangle+\frac{3Q\theta_n}{\alpha_n\eta}\|x_n-x_{n-1}\|\right],\end{aligned}$$

where $Q := \sup_{n\in\mathbb{N}}\{\|x_n-p\|, \theta\|x_n-x_{n-1}\|\} > 0$.

Claim 4. The sequence $\{\|x_n-p\|\}$ converges to zero. By Lemma 2.4, it needs to show that $\limsup_{k\to\infty}\langle Fp, p-x_{n_k+1}\rangle \leq 0$ for every subsequence $\{\|x_{n_k}-p\|\}$ of $\{\|x_n-p\|\}$ satisfying

$$\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right) \geq 0. \tag{3.16}$$

For this purpose, one assumes that $\{\|x_{n_k}-p\|\}$ is a subsequence of $\{\|x_n-p\|\}$ such that (3.16) holds. Then

$$\begin{aligned}&\liminf_{k\to\infty}\left(\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right) \\ &\quad=\liminf_{k\to\infty}\left[\left(\|x_{n_k+1}-p\|-\|x_{n_k}-p\|\right)\left(\|x_{n_k+1}-p\|+\|x_{n_k}-p\|\right)\right] \geq 0.\end{aligned}$$

By Claim 2 and the assumption on $\{\alpha_n\}$, one obtains

$$\begin{aligned}\limsup_{k\to\infty}\left(\|w_{n_k}-z_{n_k}\|^2\right) &\le \limsup_{k\to\infty}\left[\alpha_{n_k}Q_4+\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right]\\ &\le \limsup_{k\to\infty}\alpha_{n_k}Q_4+\limsup_{k\to\infty}\left[\|x_{n_k}-p\|^2-\|x_{n_k+1}-p\|^2\right]\\ &= -\liminf_{k\to\infty}\left[\|x_{n_k+1}-p\|^2-\|x_{n_k}-p\|^2\right]\le 0,\end{aligned}$$

and

$$\limsup_{k\to\infty}\left[D^{-1}\tau_{n_k}\left(\lambda^{-1}-\mu\right)\left\|r_\lambda\left(w_{n_k}\right)\right\|^2\right]^2\le 0.$$

These imply that

$$\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0 \text{ and } \lim_{k\to\infty}\tau_{n_k}\left\|r_\lambda(w_{n_k})\right\|^2=0.$$

It follows from Lemma 3.4 that $\lim_{k\to\infty}\left\|w_{n_k}-y_{n_k}\right\|=0$. Moreover, we can show that

$$\|x_{n_k+1}-z_{n_k}\|=\alpha_{n_k}\gamma\|Fz_{n_k}\|\to 0 \text{ as } n\to\infty \tag{3.17}$$

and

$$\|x_{n_k}-w_{n_k}\|=\alpha_{n_k}\cdot\frac{\theta_{n_k}}{\alpha_{n_k}}\|x_{n_k}-x_{n_k-1}\|\to 0 \text{ as } n\to\infty. \tag{3.18}$$

Combining (3.17) and (3.18), we arrive at

$$\|x_{n_k+1}-x_{n_k}\|\le\|x_{n_k+1}-z_{n_k}\|+\|z_{n_k}-w_{n_k}\|+\|w_{n_k}-x_{n_k}\|\to 0 \text{ as } n\to\infty. \tag{3.19}$$

Since the sequence $\left\{x_{n_k}\right\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$, which converges weakly to some $z\in\mathcal{H}$. By (3.18), we obtain $w_{n_k}\rightharpoonup z$ as $k\to\infty$. This together with $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0$ and Lemma 3.3 yields that $z\in\Omega$. From the assumption that p is the unique solution of the (BVIP), we deduce

$$\limsup_{k\to\infty}\left\langle Fp,p-x_{n_k}\right\rangle=\lim_{j\to\infty}\left\langle Fp,p-x_{n_{k_j}}\right\rangle=\langle Fp,p-z\rangle\le 0. \tag{3.20}$$

Using (3.19) and (3.20), we obtain

$$\limsup_{k\to\infty}\langle Fp,p-x_{n_k+1}\rangle=\limsup_{k\to\infty}\left\langle Fp,p-x_{n_k}\right\rangle\le 0. \tag{3.21}$$

From $\lim_{n\to\infty}\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|=0$ and (3.21), we observe

$$\limsup_{k\to\infty}\left[\frac{2\gamma}{\eta}\left\langle Fp,p-x_{n_k+1}\right\rangle+\frac{3Q\theta_{n_k}}{\alpha_{n_k}\eta}\|x_{n_k}-x_{n_k-1}\|\right]\le 0. \tag{3.22}$$

Combining Claim 3, Assumption (A5) and (3.22), in the light of Lemma 2.4, one concludes that $\lim_{n\to\infty}\|x_n-p\|=0$. That is, $x_n\to p$ as $n\to\infty$. This completes the proof. □

3.2 Second type of projection algorithm

Inspired by the inertial method, the Algorithm 4 proposed by Reich et al. [18] and the hybrid steepest descent method [20], we present our second iterative scheme which employs a different hyperplane from Algorithm 3.1 for solving the (BVIP). More precisely, the scheme is shown in Algorithm 3.2.

We start the convergence analysis of Algorithm 3.2 by proving the following lemma.

Algorithm 3.2 Modified inertial extragradient method for solving (BVIP).

Initialization: Take $\theta > 0$, $\ell \in (0,1)$, $\mu > 0$, $\lambda \in (0, 1/\mu)$, $\gamma \in (0, 2\beta/L_F^2)$ and let $x_0, x_1 \in \mathcal{H}$.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$, calculate x_{n+1} as follows:
Step 1. Compute $w_n = x_n + \theta_n(x_n - x_{n-1})$, where θ_n is defined in (In-Cri).
Step 2. Compute $y_n = P_C(w_n - \lambda M w_n)$. Set $r_\lambda(w_n) = w_n - y_n$.
Step 3. Compute $t_n = w_n - \tau_n r_\lambda(w_n)$, where $\tau_n = \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\left\langle M w_n - M\left(w_n - \ell^m r_\lambda(w_n)\right), r_\lambda(w_n)\right\rangle \leq \frac{\mu}{2}\|r_\lambda(w_n)\|^2. \tag{Ar-2}$$

Step 4. Compute $z_n = P_{H_n}(w_n)$, where the half-space H_n is defined by

$$H_n = \{x \in C : h_n(x) \leq 0\} \quad \text{and} \quad h_n(x) = \langle M t_n, x - w_n\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2. \tag{Hn-2}$$

Step 5. Compute $x_{n+1} = z_n - \alpha_n \gamma F z_n$.
Set $n := n+1$ and go to **Step 1**.

Lemma 3.5 *Suppose that Assumptions (A1)–(A3) hold. Let x^* be a solution of (VIP). Then $h_n(x^*) \leq 0$ and $h_n(w_n) = \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2$. In particular, if $r_\lambda(w_n) \neq 0$ then $h_n(w_n) > 0$.*

Proof From the definition of $h_n(x)$, one obtains $h_n(w_n) = \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2$. It follows from Lemma 3.2 that $\langle M t_n, x^* - t_n\rangle \leq 0$ and $\langle M w_n, r_\lambda(w_n)\rangle \geq \lambda^{-1}\|r_\lambda(w_n)\|^2$. From (Ar-2), one has

$$\langle M t_n, r_\lambda(w_n)\rangle \geq \langle M w_n, r_\lambda(w_n)\rangle - \frac{\mu}{2}\|r_\lambda(w_n)\|^2 \geq \left(\frac{1}{\lambda} - \frac{\mu}{2}\right)\|r_\lambda(w_n)\|^2,$$

which together with the definitions of $h_n(x)$ and t_n yields that

$$\begin{aligned} h_n\left(x^*\right) &= -\langle M t_n, w_n - t_n\rangle + \left\langle M t_n, x^* - t_n\right\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2 \\ &\leq -\tau_n\langle M t_n, r_\lambda(w_n)\rangle + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2 \\ &\leq -\frac{\tau_n}{2}\left(2\lambda^{-1} - \mu\right)\|r_\lambda(w_n)\|^2 + \frac{\tau_n}{2\lambda}\|r_\lambda(w_n)\|^2 \\ &= -\frac{\tau_n}{2}\left(\lambda^{-1} - \mu\right)\|r_\lambda(w_n)\|^2 \leq 0. \end{aligned}$$

This completes the proof. □

Theorem 3.2 *Assume that Assumptions (A1)–(A5) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges to the unique solution of the (BVIP) in norm.*

Proof The proof of this theorem is very similar to Theorem 3.1. To avoid repetition of expressions, we omit some details. We likewise divide this proof into four parts.
Claim 1. The sequence $\{x_n\}$ is bounded. Using the same facts as (3.6)–(3.10), we obtain that the sequences $\{x_n\}$, $\{w_n\}$, $\{y_n\}$, $\{t_n\}$, and $\{z_n\}$ are bounded.
Claim 2.

$$\|z_n - w_n\|^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n Q_4$$

and

$$\left[\frac{\tau_n}{2\lambda D}\|r_\lambda(w_n)\|^2\right]^2 \leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n Q_4.$$

As stated in Claim 2 of Theorem 3.1, we can easily follow the first inequality. Moreover, we also obtain that $h_n(x)$ is D-Lipschitz continuous on $\mathcal{H}$. From Lemmas 2.2 and 3.5, we have

$$\operatorname{dist}(w_n, H_n) \geq D^{-1} h_n(w_n) = \frac{\tau_n}{2\lambda D} \|r_\lambda(w_n)\|^2 .$$

This together with (3.6), (3.11) and (3.12) implies that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|w_n - p\|^2 - \operatorname{dist}^2(w_n, H_n) + \alpha_n Q_2 \\ &\leq \|x_n - p\|^2 - \left[\frac{\tau_n}{2\lambda D} \|r_\lambda(w_n)\|^2\right]^2 + \alpha_n Q_4,\end{aligned}$$

where Q_4 is defined in Claim 2 of Theorem 3.1. A simple transformation of the above equation can obtain the second inequality.

Claim 3.

$$\|x_{n+1} - p\|^2 \leq (1 - \alpha_n \eta) \|x_n - p\|^2 + \alpha_n \eta \left[\frac{2\gamma}{\eta} \langle Fp, p - x_{n+1}\rangle + \frac{3Q\theta_n}{\alpha_n \eta} \|x_n - x_{n-1}\|\right].$$

The result can be obtained by using the same facts as declared in Claim 3 of Theorem 3.1.

Claim 4. The sequence $\{\|x_n - p\|\}$ converges to zero. Let $\{\|x_{n_k} - p\|\}$ be a subsequence of $\{\|x_n - p\|\}$ such that (3.16) holds. It follows from Claim 2 and Assumption (A5) that

$$\limsup_{k\to\infty} \left\|w_{n_k} - z_{n_k}\right\|^2 \leq 0 \quad \text{and} \quad \limsup_{k\to\infty} \left[\frac{\tau_{n_k}}{2\lambda D} \left\|r_\lambda\left(w_{n_k}\right)\right\|^2\right]^2 \leq 0.$$

Therefore, we deduce that $\lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| = 0$ and $\lim_{k\to\infty} \tau_{n_k} \left\|r_\lambda(w_{n_k})\right\|^2 = 0$. By means of Lemma 3.4, one has $\lim_{k\to\infty} \left\|w_{n_k} - y_{n_k}\right\| = 0$. Using the same arguments as (3.17)–(3.21), we obtain

$$\limsup_{k\to\infty} \langle Fp, p - x_{n_k+1}\rangle \leq 0.$$

From Remark 3.2, Claim 3 and Lemma 2.4, we conclude that $x_n \to p$ as $n \to \infty$. The proof is now complete. □

Now, we give a special case of Theorems 3.1 and 3.2. Set $F(x) = x - f(x)$ in Algorithms 3.1 and 3.2, where mapping $f : \mathcal{H} \to \mathcal{H}$ is ρ-contraction. It can be easily verified that mapping $F : \mathcal{H} \to \mathcal{H}$ is $(1+\rho)$-Lipschitz continuous and $(1-\rho)$-strongly monotone (see [22]). In this situation, by picking $\gamma = 1$, we obtain two new inertial iterative algorithms for solving the variational inequality problem (VIP). More specifically, we have the following results.

Corollary 3.1 *Suppose that Assumptions (A1)–(A3) and (A5) hold. Let mapping $f : \mathcal{H} \to \mathcal{H}$ be ρ-contraction with $\rho \in [0, \sqrt{5} - 2)$. Take $\theta > 0$, $\ell \in (0, 1)$, $\mu > 0$ and $\lambda \in (0, 1/\mu)$. Let $x_0, x_1 \in \mathcal{H}$ be two arbitrary initial points and the iterative sequence $\{x_n\}$ be generated by the following:*

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda M w_n), \quad t_n = w_n - \tau_n (w_n - y_n), \\ x_{n+1} = \alpha_n f(z_n) + (1 - \alpha_n) z_n, \quad z_n = P_{H_n}(w_n), \\ \theta_n, \tau_n \text{ and } H_n \text{ are defined in } (In - Cri), (Ar - 1) \text{ and } (Hn - 1). \end{cases} \tag{3.23}$$

and

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}), \\ y_n = P_C (w_n - \lambda M w_n), \quad t_n = w_n - \tau_n (w_n - y_n), \\ x_{n+1} = \alpha_n f(z_n) + (1 - \alpha_n) z_n, \quad z_n = P_{H_n}(w_n), \\ \theta_n, \tau_n \text{ and } H_n \text{ are defined in } (In - Cri), (Ar - 2) \text{ and } (Hn - 2). \end{cases} \tag{3.24}$$

Then the iterative sequence $\{x_n\}$ *formed by Algorithm (3.23) (or Algorithm (3.24)) converges to* $p \in \Omega$ *in norm, where* $p = P_\Omega(f(p))$.

Remark 3.3 The iterative schemes obtained in this paper have a wide range of applications and a faster computational efficiency based on the following observations: (i) We replace the Lipschitz continuity of the mapping M in [21–24] with the uniform continuity of mapping M in the proposed algorithms. Furthermore, the suggested iterative schemes can solve the pseudomonotone (BVIP), while the algorithm stated in [21] can only solve the monotone (BVIP). (ii) Our Algorithms (3.23) and (3.24) improve many numerical methods in the literature (see, e.g., [12,13,16–19]) for solving variational inequality problems due to the fact that the mapping M involved in the proposed algorithms is pseudomonotone and uniformly continuous. (iii) Our algorithms are embedded with inertial terms making them converge faster than the algorithms without inertial [17,18] (see Sect. 4).

4 Numerical experiments and applications

In this section, we present some computational experiments to illustrate the numerical performance of the proposed algorithms over some existing ones. All the programs were implemented in MATLAB 2018a on a Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz computer with RAM 8.00 GB.

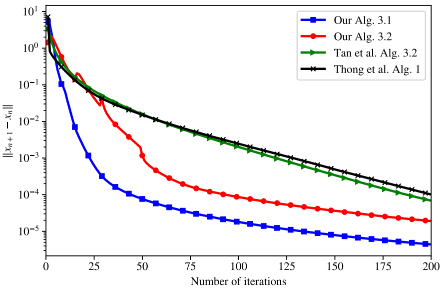
4.1 Numerical examples in finite- and infinite-dimensions

Example 4.1 We consider the following fractional programming problem:

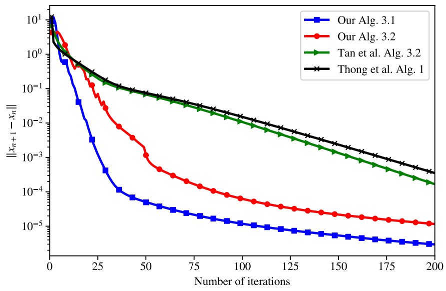
$$\min f(x) = \frac{x^\mathsf{T} Q x + a^\mathsf{T} x + a_0}{b^\mathsf{T} x + b_0}, \quad x \in C := \left\{x \in \mathbb{R}^5 : b^\mathsf{T} x + b_0 > 0\right\},$$

where Q, a, b, a_0 and b_0 are defined in [24, Example 4.1]. Let the mapping M be created by $M(x) := \nabla f(x)$. It is known that the mapping M is pseudomonotone and Lipschitz continuous (see [39]). We now consider the mapping $F : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 5)$ defined by $F(x) = Gx + q$, where $G = BB^\mathsf{T} + D + K$, and B is a $m \times m$ matrix with their entries being generated in $(0, 1)$, D is a $m \times m$ skew-symmetric matrix with their entries being generated in $(-1, 1)$, K is a $m \times m$ diagonal matrix whose diagonal entries are positive in $(0, 1)$ (so G is positive semidefinite), $q \in \mathbb{R}^m$ is a vector with entries being generated in $(0, 1)$. It is clear that F is L_F-Lipschitz continuous and β-strongly monotone with $L_F = \max\{\text{eig}(G)\}$ and $\beta = \min\{\text{eig}(G)\}$, where $\text{eig}(G)$ represents all eigenvalues of G.

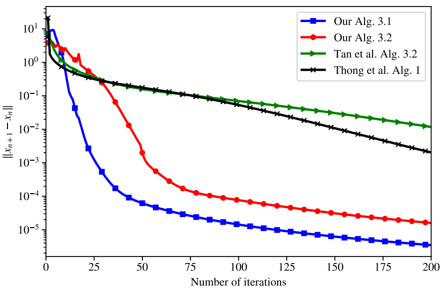
We use the proposed Algorithms 3.1 and 3.2 to solve the (BVIP) with M, F and C given above, and compare them with the Algorithm 1 introduced by Thong et al. [23] and the Algorithm 3.2 suggested by Tan, Liu and Qin [24]. The parameters of all algorithms are set as follows. Take $\alpha_n = 1/(n+1)$ and $\gamma = 1.7\beta/L_F^2$ for all algorithms. Choose $\mu = 0.1$,



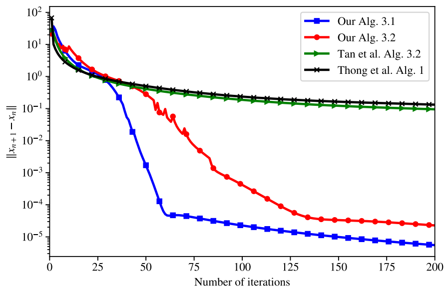
(a) $x_0 = x_1 =$ *10rand(5,1)*



(b) $x_0 = x_1 =$ *20rand(5,1)*



(c) $x_0 = x_1 =$ *40rand(5,1)*

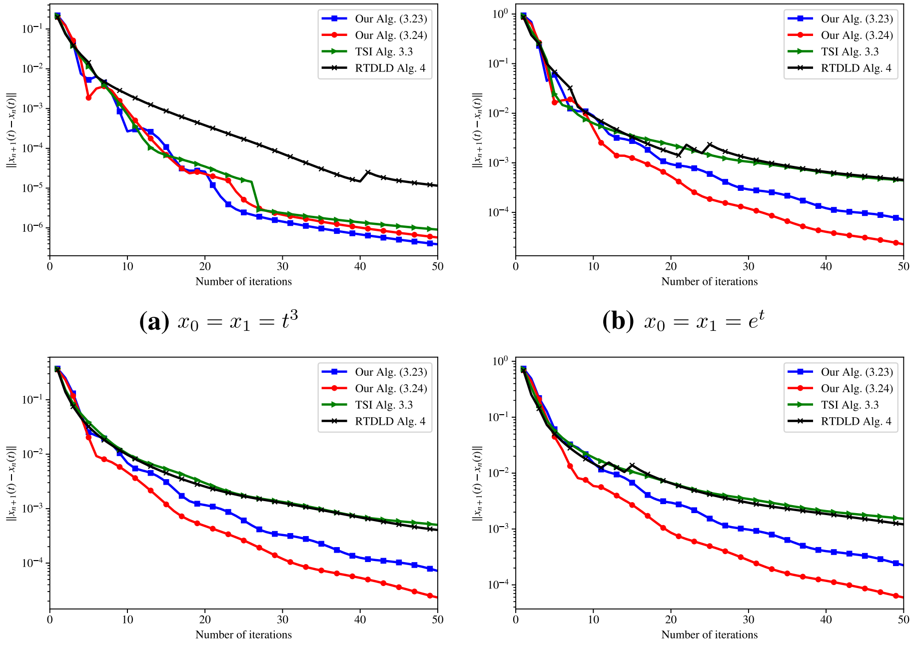


(d) $x_0 = x_1 =$ *100rand(5,1)*

Fig. 1 Numerical results of all algorithms for Example 4.1

$\lambda_1 = 0.6$ for Thong et al.'s Algorithm 1 and Tan et al.'s Algorithm 3.2. Pick $\theta = 0.4$, $\epsilon_n = 100/(n+1)^2$ for the proposed Algorithms 3.1 and 3.2, and Tan et al.'s Algorithm 3.2. Set $\phi = 1.5$ for Thong et al.'s Algorithm 1. Adopt $\ell = 0.5$, $\mu = 0.4$, $\lambda = 0.5/\mu$ for the proposed Algorithms 3.1 and 3.2. We use $D_n = \|x_{n+1} - x_n\|$ to measure the error of the nth iteration since we do not know the exact solution to the problem (BVIP). The maximum number of iterations 200 is used as a common stopping criterion for all algorithms. Numerical results of all algorithms with four different initial values $x_0 = x_1$ are reported in Fig. 1.

Example 4.2 Let $\mathcal{H} = L^2([0, 1])$ be an infinite-dimensional Hilbert space with inner product $\langle x, y\rangle = \int_0^1 x(t)y(t)\,\mathrm{d}t$ and induced norm $\|x\| = (\int_0^1 |x(t)|^2\,\mathrm{d}t)^{1/2}$. Assume that the feasible set is given by $C = \{x \in \mathcal{H} : \|x\| \leq 2\}$. Define a mapping $h : C \to \mathbb{R}$ by $h(m) = 1/(1+\|m\|^2)$. Recall that the Volterra integration operator $V : \mathcal{H} \to \mathcal{H}$ is given by $V(m)(t) = \int_0^t m(s)\,\mathrm{d}s$, $\forall t \in [0, 1], m \in \mathcal{H}$. Now, we define the mapping $M : C \to \mathcal{H}$ as follows: $M(m)(t) = h(m)V(m)(t)$, $\forall t \in [0, 1], m \in C$. Notice that the operator M is Lipschitz continuous and pseudomonotone but not monotone (see [16, Example 6.10]). We use the proposed Algorithms (3.23) and (3.24) to solve the (VIP) with M and C given above, and compare them with several previously known strongly convergent algorithms, including the Algorithm 3.3 suggested by Thong, Shehu and Iyiola [17] (shortly, TSI Alg. 3.3) and the Algorithm 4 proposed by Reich et al. [18] (shortly, RTDLD Alg. 4). The parameters of all algorithms are set as follows. Take $\alpha_n = 1/(n+1)$, $f(x) = 0.1x$, $\ell = 0.5$, $\mu = 0.4$, $\lambda = 0.5/\mu$ for all algorithms. Pick $\theta = 0.4$, $\epsilon_n = 100/(n+1)^2$ for the proposed Algorithms (3.23) and (3.24). The numerical behavior of $D_n = \|x_{n+1}(t) - x_n(t)\|$ of all algorithms with four starting points $x_0(t) = x_1(t)$ is shown in Fig. 2.



(a) $x_0 = x_1 = t^3$ **(b)** $x_0 = x_1 = e^t$

(c) $x_0 = x_1 = \cos(4t)$ **(d)** $x_0 = x_1 = \log(t)$

Fig. 2 Numerical results of all algorithms for Example 4.2

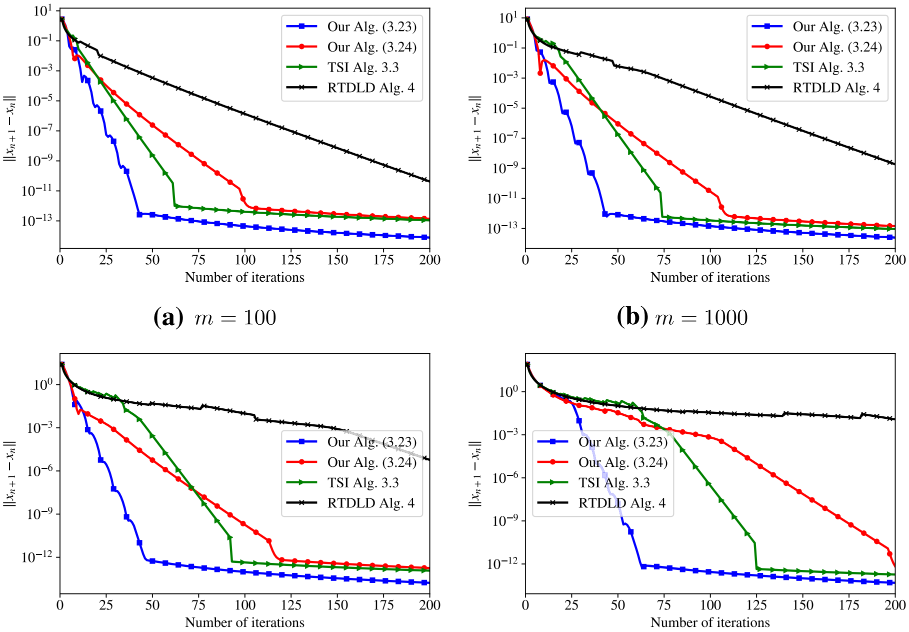
Example 4.3 Consider the Hilbert space $\mathcal{H} = l_2 := \{x = (x_1, x_2, \ldots, x_i, \ldots) \mid \sum_{i=1}^{\infty} |x_i|^2 < +\infty\}$ equipped with inner product $\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i$ and induced norm $\|x\| = \sqrt{\langle x, x\rangle}$ for any $x, y \in \mathcal{H}$. Let the feasible set by given as $C := \{x \in \mathcal{H} : |x_i| \leq 1/i\}$. Define an operator $M : C \to \mathcal{H}$ by $Mx = \big(\|x\| + 1/(\|x\| + \varphi)\big)x$ for some $\varphi > 0$. It can be verified that mapping M is pseudomonotone on $\mathcal{H}$, uniformly continuous and sequentially weakly continuous on C but not Lipschitz continuous on $\mathcal{H}$ (see [17, Example 1] for more details). In the following cases, we take $\varphi = 0.5$, and $\mathcal{H} = \mathbb{R}^m$ for different values of m. Next, we consider two different forms of operator F.

Case 1. Let mapping F be the same as defined in Example 4.1. We apply the proposed Algorithms 3.1 and 3.2 to address the problem (BVIP) with M and F given above. Choose $\alpha_n = 1/(n+1)$, $\ell = 0.5$, $\mu = 0.4$, $\lambda = 0.5/\mu$, $\gamma = 1.7\beta/L_F^2$, $\theta = 0.4$, and $\epsilon_n = 100/(n+1)^2$ for the proposed Algorithms 3.1 and 3.2. We use $D_n = \|x_{n+1} - x_n\|$ to denote the iteration error of the nth step of all algorithms, and use the maximum number of iterations 200 as a common stopping criterion. The execution time and iteration error of the proposed algorithms in four different dimensions are shown in Table 1, where "CPU" in Table 1 indicates the computation time in seconds.

Case 2. Take $F(x) = x - f(x)$, $x \in \mathcal{H}$, where mapping $f : \mathcal{H} \to \mathcal{H}$ is ρ-contraction. Now, we can use the proposed Algorithms (3.23) and (3.24) to solve the problem (VIP). We compare the proposed Algorithms (3.23) and (3.24) with the two schemes mentioned in Example 4.2 (i.e., TSI Alg. 3.3 [17] and RTDLD Alg. 4 [18]). The parameters of all algorithms are the same as in Example 4.2. The numerical performance of the sequence $\{\|x_{n+1} - x_n\|\}$ of all algorithms with four different dimensions is reported in Fig. 3.

Table 1 Numerical results of the proposed algorithms in Case 1 of Example 4.3

Algorithms	$m = 50$		$m = 200$		$m = 500$		$m = 2000$	
	D_n	CPU	D_n	CPU	D_n	CPU	D_n	CPU
Our Alg. 3.1	1.86E−09	0.0398	2.68E−11	0.0436	1.31E−12	0.0761	1.81E−12	0.8125
Our Alg. 3.2	2.92E−08	0.0309	5.84E−03	0.0394	5.60E−03	0.0712	1.33E−03	0.8355

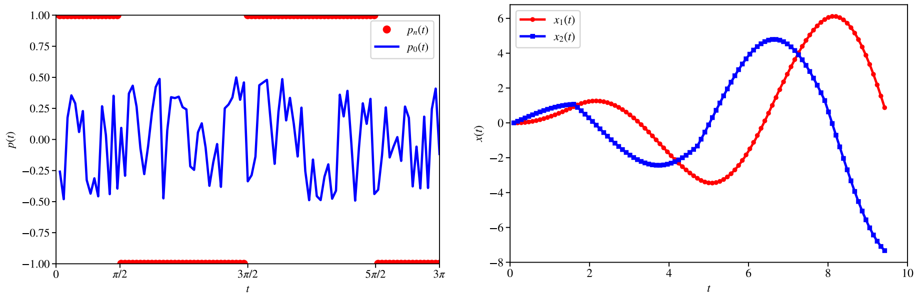


(a) $m = 100$ **(b)** $m = 1000$

(c) $m = 10000$ **(d)** $m = 100000$

Fig. 3 Numerical behavior of all algorithms in Case 2 of Example 4.3

Remark 4.1 From Examples 4.1, 4.2 and 4.3, we have the following observations: (i) From Figs. 1, 2, 3 and Table 1, it can be seen that the proposed algorithms converge quickly. Moreover, the suggested methods have a higher accuracy than the previously known ones in [17,18,23,24] under the same stopping conditions. These results are independent of the choice of initial values and the size of the dimension. Therefore, our suggested algorithms are efficient and robust. (ii) Notice that the operator M in Example 4.2 is pseudomonotone rather than monotone, and that the operator M in Example 4.3 is uniformly continuous but not Lipschitz continuous. The algorithms used in the literature (see, e.g., [12–14]) for solving monotone and Lipschitz continuous VIPs and the methods introduced in the literature (see, e.g., [21–24]) for addressing bilevel monotone (or even pseudomonotone) BVIPs will not be available in these cases. However, our proposed algorithms can work well and thus they have a broader scope of applications.



(a) Initial and optimal controls **(b)** Optimal trajectories

Fig. 4 Numerical results of the proposed Algorithm (3.23) for Example 4.4

4.2 Application to optimal control problems

Next, we use the proposed algorithms to solve the (VIP) that appears in optimal control problems. We recommend readers to refer to [15,40] for detailed description of the problem. We compare the suggested Algorithms (3.23) and (3.24) with two strongly convergent ones in the literature. Two methods used to compare here are the Algorithm (31) (shortly, TLDCR Alg. (31)) introduced by Thong et al. [23] and the Algorithm (3.39) (shortly, TLQ Alg. (3.39)) proposed by Tan, Liu and Qin [24]. The parameters of all algorithms are set as follows. Set $N = 100$, $\alpha_n = 10^{-4}/(n+1)$ for all algorithms. Take $\theta = 0.01$, $\epsilon_n = 10^{-4}/(n+1)^2$, $f(x) = 0.1x$ for the proposed Algorithms (3.23) and (3.24), and TLQ Alg. (3.39). Choose $\mu = 0.1$, $\lambda_1 = 0.4$ for TLDCR Alg. (31) and TLQ Alg. (3.39). Adopt $\alpha = 1.5$ for TLDCR Alg. (31). Pick $\ell = 0.1$, $\mu = 0.5$, $\lambda = 0.5/\mu$ for the proposed Algorithms (3.23) and (3.24). The initial controls $p_0(t) = p_1(t)$ are randomly generated in $[-1, 1]$. The stopping criterion is either $D_n = \|p_{n+1} - p_n\| \leq 10^{-3}$, or the maximum number of iterations 500000 is reached.

Example 4.4 (Control of a harmonic oscillator, see [41])

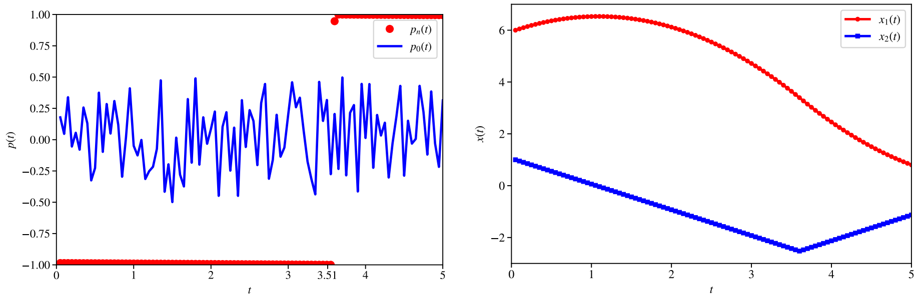
$$\begin{aligned} \text{minimize} \quad & x_2(3\pi) \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = -x_1(t) + p(t), \quad \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & p(t) \in [-1, 1]. \end{aligned}$$

The exact optimal control of Example 4.4 is known:

$$p^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2)\,; \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

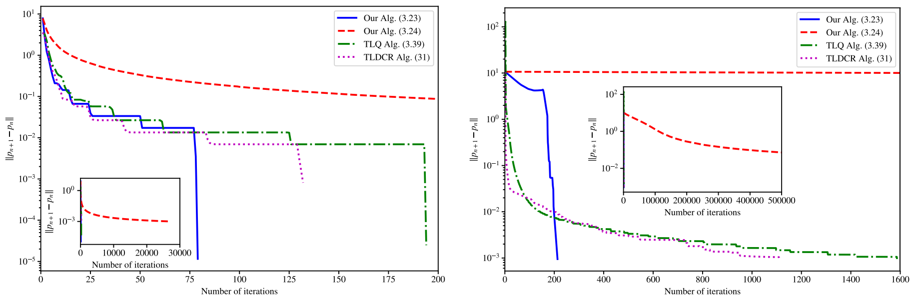
Figure 4 shows the approximate optimal control and the corresponding trajectories of the proposed Algorithm (3.23).

We now consider an example in which the terminal function is not linear.



(a) Initial and optimal controls **(b)** Optimal trajectories

Fig. 5 Numerical results of the suggested Algorithm (3.24) for Example 4.5



(a) Example 4.4 **(b)** Example 4.5

Fig. 6 Error estimates of all algorithms for Examples 4.4 and 4.5

Example 4.5 (Rocket car [40])

$$\begin{aligned} \text{minimize} \quad & 0.5\left((x_1(5))^2 + (x_2(5))^2\right), \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = p(t), \quad \forall t \in [0, 5], \\ & x_1(0) = 6, \quad x_2(0) = 1, \\ & p(t) \in [-1, 1]. \end{aligned}$$

The exact optimal control of Example 4.5 is

$$p^*(t) = \begin{cases} 1 & \text{if } t \in (3.517, 5]; \\ -1 & \text{if } t \in (0, 3.517]. \end{cases}$$

The approximate optimal control and the corresponding trajectories of the suggested Algorithm (3.24) are plotted in Fig. 5.

Finally, we compare the offered Algorithms (3.23) and (3.24) with TLDCR Alg. (31) [23] and TLQ Alg. (3.39) [24] for Examples 4.4 and 4.5. Figure 6 presents the numerical behavior of the error estimate $\|p_{n+1} - p_n\|$ with respect to the number of iterations for all algorithms. Moreover, the number of terminated iterations and the execution time of all algorithms are shown in Table 2.

Table 2 Numerical results of all algorithms for Examples 4.4 and 4.5

Algorithms	Example 4.4			Example 4.5		
	Iter.	CPU (s)	D_n	Iter.	CPU (s)	D_n
Our Alg. (3.23)	78	0.0406	1.15E−05	212	0.1364	9.48E−04
Our Alg. (3.24)	26993	10.8031	1.00E−03	500000	407.5957	7.26E−02
TLQ Alg. (3.39) [24]	193	0.1007	2.47E−05	1586	0.5253	9.50E−04
TLDCR Alg. (31) [23]	131	0.0521	8.11E−04	1109	0.3656	9.93E−04

Remark 4.2 The suggested Algorithms (3.23) and (3.24) can be applied to solve optimal control problems. As shown in Fig. 6 and Table 2, the proposed algorithms perform better when the terminal function is linear than when it is nonlinear. Furthermore, the proposed Algorithm (3.23) outperforms the existing methods in the literature [23,24]. However, it should be noted that the suggested Algorithm (3.24) needs to perform a larger number of iterations to obtain a good result (see Fig. 6 and Table 2). In future work, we will consider how to improve the convergence speed and accuracy of the suggested Algorithm (3.24).

5 Conclusions

In this paper, we proposed two projection-based algorithms to solve the bilevel variational inequality problem involving a pseudomonotone and uniformly continuous operator. Strong convergence theorems of the proposed algorithms are established without assuming Lipschitz continuity of the mapping involved. The computational efficiency of the stated algorithms is verified by means of some numerical examples in finite- and infinite-dimensional spaces and by applications in optimal control problems. The iterative schemes obtained in this paper improved and extended some existing known results in the literature for solving bilevel variational inequality problems and variational inequality problems.

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Declarations

Conflict of interest The authors declare that there have no conflict of interest.

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