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TWO NEW PROJECTION ALGORITHMS FOR VARIATIONAL INEQUALITIES IN HILBERT SPACES

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ABSTRACT. In this paper, two new projection-type algorithms are introduced for solving pseudomonotone variational inequalities in real Hilbert spaces. The proposed methods use two non-monotonic step sizes allowing them to work adaptively without the prior information of the Lipschitz constant of the operator. Strong convergence theorems for the proposed methods are established under suitable conditions. A fundamental numerical example is given to verify the efficiency of the suggested methods in comparison with some known ones.

1. Introduction

Recall that the classical variational inequality problem is formed as follows:

$$\text{(VIP)} \qquad \text{find } x^* \in \mathcal{C} \text{ such that } \langle Mx^*, x - x^* \rangle \geq 0, \quad \forall x \in \mathcal{C}\,,$$

where $\mathcal{C}$ is assumed to be a convex and closed nonempty set in a real Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and $M : \mathcal{H} \to \mathcal{H}$ is a nonlinear mapping. It is known that the theory of variational inequality provides a general and useful framework for solving problems in engineering, image processing and data sciences; see, e.g., [1–3, 8, 9, 24]. Therefore, it is useful to explore effective numerical algorithms for solving (VIP). Recently, researchers have been very interested in developing effective and robust iterative methods for solving variational inequalities; see, e.g., [11, 21, 28, 29] and the references therein. Many scholars are interested in projection-based methods among a large number of iterative schemes. It should be reminded that the extragradient method (shortly, EGM) introduced by Korpelevich [14] is a two-step projection iterative scheme. When the projection on the feasible set $\mathcal{C}$ is difficult to compute, it affects the computational efficiency of the method. Recently, some improved versions of the EGM, which require the computation of the projection on the feasible set only once in each iteration, have been proposed by researchers, see, e.g., [5, 6, 12, 18, 35]. Among these methods, this paper focuses on the subgradient extragradient method (in short, SEGM) proposed by Censor, Gibali and Reich [5]. Recall that the SEGM is described as follows.

$$\begin{cases} y_n = P_{\mathcal{C}}\left(x_n - \lambda M x_n\right)\,, \\ \mathcal{T}_n := \{x \in \mathcal{H} \mid \langle x_n - \lambda M x_n - y_n, x - y_n \rangle \leq 0\}\,, \\ x_{n+1} = P_{\mathcal{T}_n}\left(x_n - \lambda M y_n\right)\,, \quad \forall n \geq 1\,, \end{cases}$$

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where mapping M is monotone and L-Lipschitz continuous and the step size $\lambda \in (0, 1/L)$. They convert the projection of the second step in the extragradient method on the feasible set $\mathcal{C}$ to a projection on a half-space $\mathcal{T}_n$. Note that the projection on the half-space $\mathcal{T}_n$ can be calculated with a clear formula so that the SEGM only needs to evaluate the projection on the feasible set $\mathcal{C}$ once in each iteration.

It should be noted that the subgradient extragradient method obtains weak convergence in infinite-dimensional Hilbert spaces. It is known that strong convergence results in infinite-dimensional spaces are better than weak convergence results due to the need to model and analyze some practical problems in infinite-dimensional spaces, such as quantum mechanics and medical imaging. In the past decades, a large number of strongly convergent methods have been proposed to solve variational inequality problems in infinite-dimensional spaces, see, for example, Mann-type methods [25,33], viscosity-type methods [23,34,37] and projection-based methods [6,7,17]. In this paper, we focus on projection-based methods to obtain strongly convergent results. Nadezhkina and Takahashi [20] introduced a method called hybrid projection method to solve (VIP) and they established the strong convergence of the scheme. In addition, Takahashi, Takeuchi and Kubota [30] also presented a new projection-based method (now referred to as the shrinking projection algorithm) and obtained the strong convergence of the scheme.

On the other hand, notice that the SGEM needs to know the prior information about the Lipschitz constant of the mapping in order to work. However, this prior information is unknown in practical applications or estimating it requires more additional computations. Recently, some adaptive algorithms have been proposed to solve the variational inequality problem when the Lipschitz constant of the mapping is unknown, see, e.g., [33, 34, 36, 37]. It should be mentioned that the adaptive schemes proposed in [33, 34, 36, 37] may affect the computational efficiency of such algorithms due to the reason that they use a non-increasing sequence of stepsizes. Recently, Liu and Yang [16] provided several methods with non-monotonic stepsize sequences to overcome this difficulty. In addition, Cai, Dong and Peng [4] proposed an iterative scheme with a new Armijo-type stepsize to solve pseudomonotone and non-Lipschitz continuous variational inequalities, which extends the algorithms used in the literature for solving Lipschitz continuous variational inequalities. In recent years, the inertial idea has been studied by many researchers as a technique to accelerate the convergence speed of algorithms. The main feature of the inertial method is that the next iteration depends on the combination of the previous two iterations. This small change can significantly improve the computational efficiency of the algorithms without inertial terms. In recent years, many inertial-type iterative schemes have been proposed for solving variational inequalities, split feasibility problems, fixed point problems, image processing problems and other optimization problems; see, e.g., [10, 22, 26, 27, 33, 38] and the references therein.

Inspired and motivated by the above work, this paper introduces two new projection-based adaptive modified subgradient extragradient methods with inertial terms for solving pseudomonotone variational inequalities in real Hilbert spaces. Our algorithms use two new non-monotonic stepsize criteria that allow them to work adaptively. Strong convergence theorems for the proposed algorithms are established under some suitable assumptions. Finally, a primary computational test is given to support the theoretical results. This paper is organized as follows. In Section 2, we recall some preliminary results and lemmas for further use. Section 3 analyzes the convergence of the proposed algorithms. A primary numerical example is given in Section 4 to illustrate the behavior of the proposed algorithms and compare them with other ones. Finally, we conclude the paper with a brief summary in Section 5, the last section.

2. Preliminaries

In the whole paper, we use the symbol $x_n \to x$ (resp., $x_n \rightharpoonup x$) to represent the strong convergence (resp., weak convergence) of the sequence $\{x_n\}$ to x, and use $P_{\mathcal{C}} : \mathcal{H} \to \mathcal{C}$ to denote the metric projection from $\mathcal{H}$ onto $\mathcal{C}$, i.e., $P_{\mathcal{C}}(x) := \arg\min\{\|x-y\|,\, y \in \mathcal{C}\}$. It is known that $P_{\mathcal{C}}$ has the following basic properties:

$$\langle x - P_{\mathcal{C}}(x), y - P_{\mathcal{C}}(x)\rangle \leq 0,\ \forall x \in \mathcal{H}, y \in \mathcal{C}\,. \tag{2.1}$$

$$\|P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y)\|^2 \leq \langle P_{\mathcal{C}}(x) - P_{\mathcal{C}}(y), x - y\rangle\,,\ \forall x \in \mathcal{H}, y \in \mathcal{H}\,. \tag{2.2}$$

$$\|x - P_{\mathcal{C}}(x)\|^2 \leq \|x-y\|^2 - \|y - P_{\mathcal{C}}(x)\|^2\,,\ \forall x \in \mathcal{H}, y \in \mathcal{C}\,. \tag{2.3}$$

Let $\omega_w(x_n)$ denote the set of all weak limits of $\{x_n\}$, i.e., $\omega_w(x_n) := \{x \in \mathcal{H} : x_{n_j} \rightharpoonup x$ for some subsequence $\{n_j\}$ of $\{n\}\}$. Recall that a mapping $M : \mathcal{H} \to \mathcal{H}$ is called: (i) *L-Lipschitz continuous* if there is $L > 0$ such that $\|Mx - My\| \leq L\|x-y\|$, $\forall x, y \in \mathcal{H}$; (ii) *monotone* if $\langle Mx - My, x - y\rangle \geq 0$, $\forall x, y \in \mathcal{H}$; (iii) *pseudomonotone* if $\langle Mx, y-x\rangle \geq 0 \Longrightarrow \langle My, y-x\rangle \geq 0$, $\forall x, y \in \mathcal{H}$; (iv) *sequentially weakly continuous*, if for any sequence $\{x_n\}$ converges weakly to a point $x \in \mathcal{H}$, then $\{Mx_n\}$ weakly converges to Mx.

The following lemmas are useful for the convergence analysis of our main results.

Lemma 2.1 ([13])**.** *Let $\mathcal{C}$ be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}$. Given $x, y, z \in \mathcal{H}$ and $a \in \mathbb{R}$. $\{v \in \mathcal{C} : \|y-v\|^2 \leq \|x-v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

Lemma 2.2 ([19])**.** *Let $\mathcal{C}$ be a closed convex subset of a real Hilbert space $\mathcal{H}$, $\{x_n\} \subset \mathcal{H}$ and $u \in \mathcal{H}$. Let $q = P_{\mathcal{C}}(u)$. If $\omega_w(x_n) \subset \mathcal{C}$ and $\|x_n - u\| \leq \|u - q\|$, $\forall n \in \mathbb{N}$. Then $x_n \to q$ as $n \to \infty$.*

3. Main results

In this section, we introduce two new numerical algorithms for solving (VIP) in real Hilbert spaces. These two schemes are inspired by the inertial method, the subgradient extragradient method and the shrinking projection method. First, a new iterative scheme with a non-monotonic step size criterion is given in Algorithm 3.1. We assume that the suggested Algorithm 3.1 satisfies the following conditions.

- (C1) The feasible set $\mathcal{C}$ is nonempty closed and convex. The solution set of (VIP) is denoted as $\mathrm{VI}(\mathcal{C}, M)$ and is assumed to be nonempty.
- (C2) The mapping $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, L-Lipschitz continuous on $\mathcal{H}$ and sequentially weakly continuous on $\mathcal{C}$.

The Algorithm 3.1 is formulated as follows.

Algorithm 3.1 The modified SEGM with self-adaptive stepsizes

Initialization: Take $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$, $\lambda_1 > 0$, $\mu \in (0,1)$, $\beta \in (1/(2-\mu), 1/\mu)$ and $\mathcal{C}_1 = \mathcal{H}$. Choose a nonnegative real sequence $\{\xi_n\}$ such that $\sum_{n=1}^{\infty} \xi_n < +\infty$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.

Iterative Steps: Given x_{n-1} and x_n $(n \geq 1)$. Calculate x_{n+1} as follows:

Step 1. Compute $u_n = x_n + \theta_n (x_n - x_{n-1})$.

Step 2. Compute

$$y_n = P_{\mathcal{C}} (u_n - \beta \lambda_n M u_n) .$$

If $y_n = u_n$ then stop and y_n is a solution of (VIP). Otherwise, go to *Step 3.*

Step 3. Compute

$$z_n = P_{\mathcal{T}_n} (u_n - \lambda_n M y_n) ,$$

where $\mathcal{T}_n := \{x \in \mathcal{H} \mid \langle u_n - \beta\lambda_n M u_n - y_n, x - y_n \rangle \leq 0\}$. Update λ_{n+1} by

$$\lambda_{n+1} = \begin{cases} \min\left\{\mu \dfrac{\|u_n - y_n\|^2 + \|z_n - y_n\|^2}{2\langle M u_n - M y_n, z_n - y_n\rangle}, \lambda_n + \xi_n\right\}, & \text{if } \langle M u_n - M y_n, z_n - y_n\rangle > 0, \\ \lambda_n + \xi_n, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 4. Compute $x_{n+1} = P_{\mathcal{C}_{n+1}}(x_0)$, where $\mathcal{C}_{n+1}$ is defined by

$$\mathcal{C}_{n+1} := \left\{p \in \mathcal{C}_n : \|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^{\dagger}(\|y_n - u_n\|^2 + \|z_n - y_n\|^2)\right\},$$

where $\beta^{\dagger} = 2 - \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}$ when $\beta \in (0,1]$ and $\beta^{\dagger} = \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}$ when $\beta > 1$.

Set $n := n+1$ and go to *Step 1.*

The following lemmas are useful in the convergence analysis of Algorithm 3.1.

Lemma 3.1 ([16])**.** *Suppose that Condition (C2) holds. Then the sequence* $\{\lambda_n\}$ *generated by* (3.1) *is well defined and* $\lim_{n\to\infty} \lambda_n = \lambda$ *and* $\lambda \in \left[\min\{\mu/L, \lambda_1\}, \lambda_1 + \sum_{n=1}^{\infty} \xi_n\right]$.

Proof. The proof is very similar to Lemma 3.1 in [16]. So we omit the details. □

Lemma 3.2. *Suppose that Condition (C2) holds. Let* $\{z_n\}$ *be a sequence generated by Algorithm 3.1. Then, for all* $p \in \mathrm{VI}(\mathcal{C}, M)$,

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^{\dagger}\left(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\right),$$

where $\beta^{\dagger} = 2 - \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}$ *when* $\beta \in (0,1]$ *and* $\beta^{\dagger} = \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}$ *when* $\beta > 1$.

Proof. From the definition of z_n and the property of projection (2.3), we have

$$\begin{aligned} \|z_n - p\|^2 &= \|P_{\mathcal{T}_n}(u_n - \lambda_n M y_n) - p\|^2 \\ &\leq \|u_n - \lambda_n M y_n - p\|^2 - \|u_n - \lambda_n M y_n - z_n\|^2 \\ &= \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\langle \lambda_n M y_n, z_n - p\rangle \\ &= \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\langle \lambda_n M y_n, z_n - y_n\rangle \\ &\quad - 2\langle \lambda_n M y_n, y_n - p\rangle . \end{aligned} \tag{3.2}$$

Since $p \in \mathrm{VI}(\mathcal{C}, M)$ and $y_n \in \mathcal{C}$, we have $\langle Mp, y_n - p\rangle \geq 0$. By the pseudomontonicity of mapping M, we get $\langle My_n, y_n - p\rangle \geq 0$. Thus, (3.2) reduces to

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \|u_n - z_n\|^2 - 2\left\langle \lambda_n M y_n, z_n - y_n\right\rangle . \tag{3.3}$$

Now we estimate $2\left\langle \lambda_n M y_n, z_n - y_n\right\rangle$. Note that

$$-\|u_n - z_n\|^2 = -\|u_n - y_n\|^2 - \|y_n - z_n\|^2 + 2\left\langle u_n - y_n, z_n - y_n\right\rangle . \tag{3.4}$$

In addition,

$$\begin{aligned}
&\left\langle u_n - y_n, z_n - y_n\right\rangle \\
&= \left\langle u_n - y_n - \beta\lambda_n M u_n + \beta\lambda_n M u_n - \beta\lambda_n M y_n + \beta\lambda_n M y_n, z_n - y_n\right\rangle \\
&= \left\langle u_n - \beta\lambda_n M u_n - y_n, z_n - y_n\right\rangle \\
&\quad + \beta\lambda_n \left\langle M u_n - M y_n, z_n - y_n\right\rangle + \left\langle \beta\lambda_n M y_n, z_n - y_n\right\rangle .
\end{aligned} \tag{3.5}$$

Since $z_n \in \mathcal{T}_n$, one gets

$$\left\langle u_n - \beta\lambda_n M u_n - y_n, z_n - y_n\right\rangle \leq 0\,. \tag{3.6}$$

According to the definition of λ_n, it is easy to obtain

$$\left\langle M u_n - M y_n, z_n - y_n\right\rangle \leq \frac{\mu}{2\lambda_{n+1}}\|u_n - y_n\|^2 + \frac{\mu}{2\lambda_{n+1}}\|z_n - y_n\|^2 . \tag{3.7}$$

Substituting (3.5), (3.6) and (3.7) into (3.4), we get

$$-\|u_n - z_n\|^2 \leq -\Big(1 - \frac{\beta\mu\lambda_n}{\lambda_{n+1}}\Big)\Big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\Big) + 2\beta\left\langle \lambda_n M y_n, z_n - y_n\right\rangle ,$$

which implies that

$$-2\left\langle \lambda_n M y_n, z_n - y_n\right\rangle \leq -\Big(\frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\Big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\Big) + \frac{1}{\beta}\|u_n - z_n\|^2 .$$

This together with (3.3) concludes that

$$\begin{aligned}
\|z_n - p\|^2 \leq \|u_n - p\|^2 &- \Big(\frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\Big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\Big) \\
&- \Big(1 - \frac{1}{\beta}\Big)\|u_n - z_n\|^2 .
\end{aligned} \tag{3.8}$$

Note that

$$\|u_n - z_n\|^2 \leq 2\big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\big)\,,$$

which yields that

$$-\Big(1 - \frac{1}{\beta}\Big)\|u_n - z_n\|^2 \leq -2\Big(1 - \frac{1}{\beta}\Big)\Big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\Big), \ \ \forall \beta \in (0, 1]\,.$$

This together with (3.8) implies

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \Big(2 - \frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\Big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\Big), \ \ \forall \beta \in (0, 1]\,.$$

On the other hand, if $\beta > 1$, then we get

$$\|z_n - p\|^2 \leq \|u_n - p\|^2 - \Big(\frac{1}{\beta} - \frac{\mu\lambda_n}{\lambda_{n+1}}\Big)\Big(\|u_n - y_n\|^2 + \|z_n - y_n\|^2\Big), \ \ \forall \beta > 1\,.$$

This completes the proof of the lemma. □

Remark 3.3. From Lemma 3.1 and the assumptions of the parameters μ and β (i.e., $\mu \in (0,1)$ and $\beta \in (1/(2-\mu), 1/\mu)$), we can obtain that $\beta^{\dagger} > 0$ for all $n \geq n_0$ in Lemma 3.2 always holds.

Theorem 3.4. *Suppose Conditions (C1) and (C2) hold. Then the sequence $\{x_n\}$ created by Algorithm 3.1 converges to $q \in \mathrm{VI}(\mathcal{C}, M)$ in norm, where $q = P_{\mathrm{VI}(\mathcal{C},M)}(x_0)$.*

Proof. We divide our proof in three steps. To begin with, our first goal is to show that $\mathrm{VI}(\mathcal{C}, M) \subset \mathcal{C}_{n+1}$ for all $n \geq 1$. Indeed, it follows from Lemma 2.1 that $\mathcal{C}_{n+1}$ is convex and closed. Moreover, one has $p \in \mathcal{C}_{n+1}$ by means of Lemma 3.2. Thus, we get $\mathrm{VI}(\mathcal{C}, M) \subset \mathcal{C}_{n+1} \subset \mathcal{C}_n$ for all $n \geq 1$. So $x_{n+1} = P_{\mathcal{C}_{n+1}}(x_0)$ is well defined. The next thing to do in the proof is show that $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$ and $\lim_{n\to\infty} \|y_n - u_n\| = 0$. From $x_n = P_{\mathcal{C}_n}(x_0)$ and $\mathrm{VI}(\mathcal{C}, M) \subset \mathcal{C}_n$, we obtain $\|x_n - x_0\| \leq \|z - x_0\|$, $\forall z \in \mathrm{VI}(\mathcal{C}, M)$. In particular, one has

$$\|x_n - x_0\| \leq \|q - x_0\|\,, \quad \text{where } q = P_{\mathrm{VI}(\mathcal{C},M)}(x_0)\,. \tag{3.9}$$

This implies that the sequence $\{x_n\}$ is bounded. We get that the sequences $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ are also bounded. Since $x_n = P_{\mathcal{C}_n}(x_0)$ and $x_{n+1} \subset \mathcal{C}_n$, one obtains $\|x_n - x_0\| \leq \|x_{n+1} - x_0\|$, which indicates that $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Using the property of projection (2.3), one has

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2\,, \quad \forall n \geq 1\,. \tag{3.10}$$

Combining (3.9) and (3.10), we get

$$\sum\nolimits_{n=1}^{N} \|x_{n+1} - x_n\|^2 \leq \|q - x_0\|^2 - \|x_1 - x_0\|^2\,,$$

which yields that $\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2$ is convergent. Thus, $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$. From the definition of u_n, it follows that

$$\|u_n - x_n\| = |\theta_n|\,\|x_n - x_{n-1}\| \leq |\theta|\,\|x_n - x_{n-1}\| \to 0, \text{ as } n \to \infty\,.$$

Therefore, we get that $\lim_{n\to\infty} \|u_n - x_{n+1}\| = 0$. From $x_{n+1} \in \mathcal{C}_{n+1}$ and the definiton of $\mathcal{C}_{n+1}$, one infers that $\|z_n - x_{n+1}\|^2 \leq \|u_n - x_{n+1}\|^2$. This together with $\lim_{n\to\infty} \|u_n - x_{n+1}\| = 0$ yields that $\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0$ and $\|z_n - u_n\| = 0$. In addition, it follows from Lemma 3.2 and Remark 3.3 that

$$\beta^{\dagger}\big(\,\|y_n - u_n\|^2 + \|z_n - y_n\|^2\,\big) \leq \big(\,\|z_n - p\| + \|u_n - p\|\,\big)\|z_n - u_n\| \to 0, \text{ as } n \to \infty\,.$$

This implies that $\lim_{n\to\infty} \|y_n - u_n\| = 0$ and $\lim_{n\to\infty} \|z_n - y_n\| = 0$. Finally, we need to show that the sequence $\{x_n\}$ converges to $q \in \mathrm{VI}(\mathcal{C}, M)$ in norm. From [32, Lemma 3.3], it can be easily seen that each sequential weak cluster point of $\{x_n\}$ is in $\mathrm{VI}(\mathcal{C}, M)$, i.e., $\omega_w(x_n) \subset \mathrm{VI}(\mathcal{C}, M)$. This together with (3.9), in the light of Lemma 2.2, yields that $x_n \to q$ as $n \to \infty$. The proof is completed. □

Next, we present an iterative scheme (see Algorithm 3.2) for solving (VIP) with a pseudomonotone and non-Lipschitz continuous operator. In our Algorithm 3.2, we replace the condition (C2) in Algorithm 3.1 with the following condition (C3).

(C3) The operator $M : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, uniformly continuous on $\mathcal{H}$ and sequentially weakly continuous on $\mathcal{C}$.

Now we are ready to describe the proposed Algorithm 3.2.

Algorithm 3.2 The modified SEGM with Armijo-like stepsizes

Initialization: Take $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$, $\delta > 0$, $\ell \in (0,1)$, $\mu \in (0,1)$, $\beta \in (1/(2-\mu), 1/\mu)$ and $\mathcal{C}_1 = \mathcal{H}$. Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given x_{n-1} and x_n $(n \geq 1)$. Calculate x_{n+1} as follows:
Step 1. Compute $u_n = x_n + \theta_n (x_n - x_{n-1})$.
Step 2. Compute

$$y_n = P_{\mathcal{C}} (u_n - \beta\lambda_n M u_n) \ .$$

If $y_n = u_n$ then stop and y_n is a solution of (VIP). Otherwise, go to *Step 3.*
Step 3. Compute

$$z_n = P_{\mathcal{T}_n} (u_n - \lambda_n M y_n) \ ,$$

where $\mathcal{T}_n := \{x \in \mathcal{H} \mid \langle u_n - \beta\lambda_n M u_n - y_n, x - y_n\rangle \leq 0\}$, $\lambda_n := \delta\ell^{m_n}$ and m_n is the smallest nonnegative integer m satisfying

$$\delta\ell^m \langle My_n - Mu_n, y_n - z_n\rangle \leq \frac{\mu}{2} \left[\|u_n - y_n\|^2 + \|y_n - z_n\|^2\right] \ . \tag{3.11}$$

Step 4. Compute $x_{n+1} = P_{\mathcal{C}_{n+1}}(x_0)$, where $\mathcal{C}_{n+1}$ is defined by

$$\mathcal{C}_{n+1} := \left\{p \in \mathcal{C}_n : \|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^\dagger(\|y_n - u_n\|^2 + \|z_n - y_n\|^2)\right\},$$

where $\beta^\dagger = 2 - \frac{1}{\beta} - \mu$ when $\beta \in (0,1]$ and $\beta^\dagger = \frac{1}{\beta} - \mu$ when $\beta > 1$.
Set $n := n+1$ and go to *Step 1.*

Remark 3.5. Suppose Condition (C3) holds. Let $\{u_n\}$ and $\{y_n\}$ be two sequences created by Algorithm 3.2. Following the proof of Lemma 3.1 in [32], we can get that the Armijo criterion (3.11) is well defined. Moreover, by [4, Lemma 3.2], it can be easily seen that each sequential weak cluster point of $\{x_n\}$ is in $\mathrm{VI}(\mathcal{C}, M)$.

Lemma 3.6. *Assume that Condition (C3) holds. Let $\{z_n\}$ be a sequence formed by Algorithm 3.2. Then, $\|z_n - p\|^2 \leq \|u_n - p\|^2 - \beta^\dagger(\|u_n - y_n\|^2 + \|z_n - y_n\|^2)$, $\forall p \in \mathrm{VI}(\mathcal{C}, M)$, where $\beta^\dagger = 2 - \frac{1}{\beta} - \mu$ when $\beta \in (0,1]$ and $\beta^\dagger = \frac{1}{\beta} - \mu$ when $\beta > 1$.*

Proof. The proof follows the proof of Lemma 3.2 and thus it is omitted. ☐

Theorem 3.7. *Assume Conditions (C1) and (C3) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges to $q \in \mathrm{VI}(\mathcal{C}, M)$ in norm, where $q = P_{\mathrm{VI}(\mathcal{C},M)}(x_0)$.*

Proof. The proof of this result follows almost in the same way as that of Theorem 3.4 but we apply Lemma 3.6 in place of Lemma 3.2. ☐

Remark 3.8. We have the following observations for the proposed algorithms.

(1) Note that the inertial parameter θ_n in the suggested Algorithms 3.1 and 3.2 are located in $[-\theta, \theta]$ for some $\theta > 0$. They are weaker than some known inertial-type methods in the literature (see, e.g., [27, 33, 36, 37]).
(2) In our Algorithms 3.1 and 3.2, y_n is computed as $y_n = P_{\mathcal{C}}(u_n - \beta\lambda_n M u_n)$. Notice that the step size for computing y_n is $\beta\lambda_n$, which is different from the subgradient extragradient method introduced by Censor et al. [5]. Our main

numerical example shows that this modification can improve the convergence speed of the proposed algorithms (see Section 4).

(3) The algorithms presented in this paper can solve pseudomonotone variational inequality problems, which extend the methods used in the literature for solving monotone variational inequalities (see, e.g., [11, 18, 28, 33, 37]). On the other hand, note that the proposed Algorithm 3.2 requires the mapping M to be uniformly continuous rather than Lipschitz continuous, while the algorithm presented in [11, 25, 28, 33, 37] can only solve variational inequalities with a Lipschitz continuous operator. Moreover, the step size in the suggested Algorithm 3.1 is non-monotonic, i.e., the step size is allowed to increase with iteration. Thus, the iterative schemes stated in this paper have a wider range of applications and a faster convergence speed.

4. A fundamental numerical example

In this section, we implement a computational example to illustrate the numerical performance of the presented methods and compare them with some known ones in the literature [15, 31]. All the programs were implemented in MATLAB 2018a on a personal computer with RAM 8.00 GB.

First, we state the algorithms that need to be compared in [15, 31] as follows.

Theorem 4.1 ([15]). *Suppose that Conditions (C1) and (C2) hold. Take $\theta_n \in (0, +\infty)$, $\delta \in (0, 1)$, $\ell \in (0, 1)$ and $\mu \in (0, 1/\sqrt{2})$. Let $\{x_n\}$ be a sequence created by*

$$\begin{cases} u_n = x_n + \theta_n (x_n - x_{n-1}), \, y_n = P_{\mathcal{C}} (x_n - \lambda_n M u_n), \\ z_n = y_n - \lambda_n (M y_n - M u_n), \, x_{n+1} = P_{\mathcal{C}_n \cap \mathcal{Q}_n} (x_0), \\ \mathcal{C}_n := \big\{ p \in \mathcal{H} : \|z_n - p\|^2 \le \|x_n - p\|^2 \\ \qquad - \left(1 - 2\mu^2\right) \|x_n - y_n\|^2 + 2\mu^2 \theta_n^2 \|x_{n-1} - x_n\|^2 \big\}, \\ \mathcal{Q}_n := \big\{ p \in \mathcal{H} : \langle x_n - p, x_n - x_0 \rangle \le 0 \big\}, \end{cases}$$

where $\lambda_n = \delta \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying $\mu \|y_n - u_n\| \ge \delta \ell^m \|M y_n - M u_n\|$. Then the sequence $\{x_n\}$ converges to $q = P_{\mathrm{VI}(\mathcal{C}, M)}(x_0)$ in norm.

Theorem 4.2 ([31]). *Assume that Conditions (C1) and (C2) hold. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = x_n + \theta_n (x_n - x_{n-1}), \, y_n = P_{\mathcal{C}} (u_n - \lambda_n M u_n), \\ \mathcal{T}_n := \{ x \in \mathcal{H} \mid \langle u_n - \lambda_n M u_n - y_n, x - y_n \rangle \le 0 \}, \\ z_n = P_{\mathcal{T}_n} (u_n - \lambda_n M y_n), \, x_{n+1} = P_{\mathcal{C}_{n+1}} (x_0), \end{cases}$$

where $\theta_n \in [-\theta, \theta]$ for some $\theta > 0$, the next step size λ_{n+1} is updated by

$$\lambda_{n+1} = \begin{cases} \min \left\{ \dfrac{\mu \|u_n - y_n\|}{\|M u_n - M y_n\|}, \lambda_n \right\}, & \text{if } M u_n - M y_n \ne 0; \\ \lambda_n, & \text{otherwise.} \end{cases} \tag{4.1}$$

where $\lambda_1 > 0$, $\mu \in (0, 1)$, and $\mathcal{C}_{n+1}$ is defined as follows

$$\mathcal{C}_{n+1} := \Big\{ p \in \mathcal{C}_n : \|z_n - p\|^2 \le \|u_n - p\|^2 - \Big(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\Big) \big(\|y_n - u_n\|^2 + \|z_n - y_n\|^2\big) \Big\}.$$

Then the sequence $\{x_n\}$ *converges to* $q = P_{\mathrm{VI}(\mathcal{C},M)}(x_0)$ *in norm.*

Theorem 4.3 ([31]). *Suppose that Conditions (C1) and (C2) hold. Let* $\{x_n\}$ *be a sequence formed by*

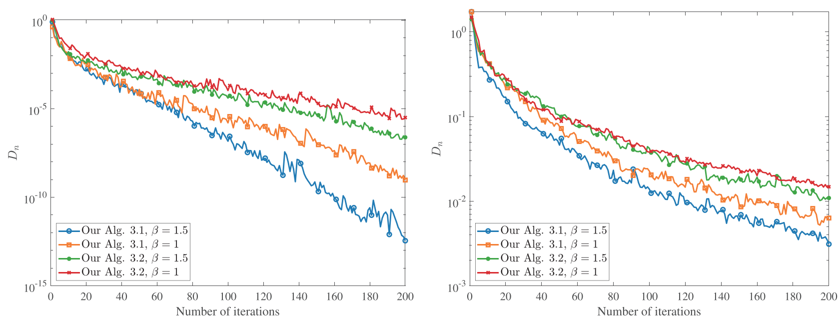
$$\begin{cases} u_n = x_n + \theta_n (x_n - x_{n-1}), \; y_n = P_{\mathcal{C}} (u_n - \lambda_n M u_n), \\ z_n = y_n - \lambda_n (M y_n - M u_n), \; x_{n+1} = P_{\mathcal{C}_{n+1}}(x_0), \end{cases}$$

where $\theta_n \in [-\theta, \theta]$ *for some* $\theta > 0$, *the next step size* λ_{n+1} *is updated by* (4.1) *and* $\mathcal{C}_{n+1}$ *is defined as follows*

$$\mathcal{C}_{n+1} := \left\{ p \in \mathcal{C}_n : \|z_n - p\|^2 \leq \|u_n - p\|^2 - \left(1 - \frac{\mu^2 \lambda_n^2}{\lambda_{n+1}^2}\right) \|u_n - y_n\|^2 \right\}.$$

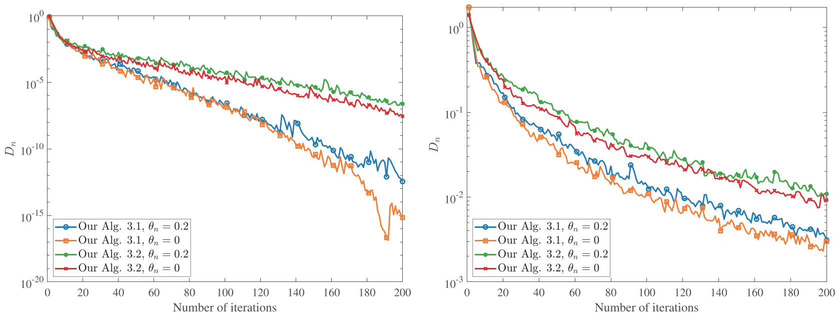
Then the sequence $\{x_n\}$ *converges to* $q = P_{\mathrm{VI}(\mathcal{C},M)}(x_0)$ *in norm.*

Example 4.4. We consider the form of linear operator $M : \mathbb{R}^m \to \mathbb{R}^m$ $(m = 5, 10)$ as follows: $M(x) = Gx + g$, where $g \in \mathbb{R}^m$ and $G = BB^{\mathsf{T}} + D + E$, $B \in \mathbb{R}^{m \times m}$, $D \in \mathbb{R}^{m \times m}$ is skew-symmetric matrix and $E \in \mathbb{R}^{m \times m}$ is diagonal matrix whose diagonal terms are non-negative (hence G is positive symmetric definite). We choose the feasible set as $\mathcal{C} = \{x \in \mathbb{R}^m : -2 \leq x_i \leq 5, \, i = 1, \ldots, m\}$. It is known that mapping M is monotone and Lipschitz continuous with constant $L = \|G\|$. In this example, both B, D entries are randomly created in $[-2, 2]$, E is generated randomly in $[0, 2]$ and $g = \mathbf{0}$. It can be easily seen that the solution to the problem is $x^* = \{\mathbf{0}\}$. We compare the porposed algorithms with the Algorithm 1 suggested by Liu and Qin [15] (shortly, LQ Alg. 1) and the Algorithms 3.1 and 3.2 presented by Tan and Cho [31] (shortly, TC Alg. 3.1 and TC Alg. 3.2). Take $\theta_n = 0.2$ for all algorithms. Choose $\lambda_1 = 0.05$ and $\mu = 0.4$ for the proposed Algorithm 3.1, TC Alg. 3.1 and TC Alg. 3.2. Set $\xi_n = 1/(n+1)^{1.1}$ for the suggested Algorithm 3.1. Pick $\delta = 0.05$, $\ell = 0.5$ and $\mu = 0.4$ for the proposed Algorithm 3.2 and LQ Alg. 1. The function $D_n = \|x_n - x^*\|^2$ is used to measure the calculation error of all algorithms in the n-th step. The maximum number of iterations 400 as a common stopping criterion. The numerical behaviors of the proposed algorithms with different parameters β and θ_n are shown in Fig. 1 and Fig. 2, respectively. Numerical results of all algorithms with two dimensions are reported in Fig. 3.



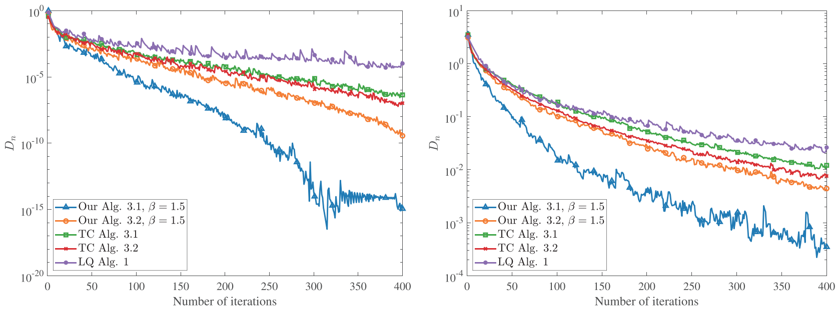
(a) $m = 5$ (b) $m = 10$

FIGURE 1. The behavior of our algorithms ($\theta_n = 0.2$) with different β



(a) $m = 5$ (b) $m = 10$

FIGURE 2. The behavior of our algorithms ($\beta = 1.5$) with different θ_n



(a) $m = 5$ (b) $m = 10$

FIGURE 3. Numerical results of all algorithms

Remark 4.5. From Fig. 1, it can be seen that the proposed Algorithm 3.1 and Algorithm 3.2 converge faster at $\beta = 1.5$ than they do at $\beta = 1$. This means that our algorithms can achieve a faster convergence speed when the parameter β is chosen at a suitable value. Moreover, our algorithms converge faster than the iterative schemes presented in the literature [15,31], and these results are independent of the size of the dimension and the choice of the initial values (see Fig. 3). Thus, the algorithms introduced in this paper are useful, efficient and robust. On the other hand, the information expressed in Fig. 2 shows that the addition of the inertial term has no positive effect on the algorithms proposed in this paper. However, this is only a fundamental numerical example. It will be our future work about the reasons for the generation of this phenomenon.

5. CONCLUSIONS

In this paper, we presented two projection-based methods to solve variational inequalities in infinite-dimensional Hilbert spaces. The proposed methods are inspired by the inertial method, the subgradient extragradient method and the shrinking projection method. The strong convergence of the iterative sequences generated by the presented algorithms is established without requiring the prior knowledge of the Lipschitz constant of the mapping. Moreover, our algorithms apply two new non-monotonic stepsize sequences, which makes them have a faster convergence speed than some known ones in the literature. Finally, we give a basic numerical example to verify the advantages and computational efficiency of the proposed methods.

References

[1] N. T. An, P. D. Dong and X. Qin, *Robust feature selection via nonconvex sparsity-based methods*, J. Nonlinear Var. Anal. **5** (2021), 59–77.

[2] N. T. An, N. M. Nam and X. Qin, *Solving k-center problems involving sets based on optimization techniques*, J. Global Optim. **76** (2020), 189–209.

[3] Q. H. Ansari, M. Islam and J. C. Yao, *Nonsmooth variational inequalities on Hadamard manifolds*, Appl. Anal. **99** (2020), 340–358.

[4] G. Cai, Q. L. Dong and Y. Peng, *Strong convergence theorems for solving variational inequality problems with pseudo-monotone and non-Lipschitz Operators*, J. Optim. Theory Appl. **188** (2021), 447–472.

[5] Y. Censor, A. Gibali and S. Reich, *The subgradient extragradient method for solving variational inequalities in Hilbert space*, J. Optim. Theory Appl. **148** (2011), 318–335.

[6] Y. Censor, A. Gibali and S. Reich, *Strong convergence of subgradient extragradient methods for the variational inequality problem in Hilbert space*, Optim. Methods Softw. **26** (2011), 827–845.

[7] L. C. Ceng, *A subgradient-extragradient method for bilevel equilibrium problems with the constraints of variational inclusion systems and fixed point problems*, Commun. Optim. Theory **2021** (2021): Article ID 4.

[8] T. H. Cuong, J. C. Yao and N. D. Yen, *Qualitative properties of the minimum sum-of-squares clustering problem*, Optimization **69** (2020), 2131–2154.

[9] P. Cubiotti and J. C. Yao, *On the Cauchy problem for a class of differential inclusions with applications*, Appl. Anal. **99** (2020), 2543–2554.

[10] J. Fan, L. Liu and X. Qin, *A subgradient extragradient algorithm with inertial effects for solving strongly pseudomonotone variational inequalities*, Optimization **69** (2020), 2199–2215.

[11] A. Gibali and Y. Shehu, *An efficient iterative method for finding common fixed point and variational inequalities in Hilbert spaces*, Optimization **68** (2019), 13–32.

[12] B. S. He, *A class of projection and contraction methods for monotone variational inequalities*, Appl. Math. Optim. **35** (1997), 69–76.

[13] T. H. Kim and H. K. Xu, *Strong convergence of modified Mann iterations for asymptotically nonexpansive mappings and semigroups*, Nonlinear Anal. **64** (2006), 1140–1152.

[14] G. M. Korpelevich, *The extragradient method for finding saddle points and other problems*, Èkonom. i Mat. Metody **12** (1976), 747–756.

[15] L. Liu and X. Qin, *Strong convergence of an extragradient-like algorithm involving pseudomonotone mappings*, Numer. Algorithms **83** (2020), 1577–1590.

[16] H. Liu and J. Yang, *Weak convergence of iterative methods for solving quasimonotone variational inequalities*, Comput. Optim. Appl. **77** (2020), 491–508.

[17] Z. L. Ma, L. Wang, S. F. Zou and X. Sun, *A hybrid projection algorithm for a split equality problem in Banach spaces*, J. Nonlinear Funct. Anal. **2021** (2021): Article ID 22.

[18] Y. Malitsky, *Projected reflected gradient methods for monotone variational inequalities*, SIAM J. Optim. **25** (2015), 502–520.

[19] C. Martinez-Yanes and H.K. Xu, *Strong convergence of the CQ method for fixed point iteration processes*, Nonlinear Anal. **64** (2006), 2400–2411.

[20] N. Nadezhkina and W. Takahashi, *Strong convergence theorem by a hybrid method for nonexpansive mappings and Lipschitz-continuous monotone mappings*, SIAM J. Optim. **16** (2006), 1230–1241.

[21] L. V. Nguyen, Q. H. Ansari and X. Qin, *Weak sharpness and finite convergence for solutions of nonsmooth variational inequalities in Hilbert spaces*, Appl. Math. Optim. **84** (2021), 807–828.

[22] X. Qin, L. Wang and J. C. Yao, *Inertial splitting method for maximal monotone mappings*, J. Nonlinear Convex Anal. **21** (2020), 2325–2333.

[23] X. Qin and J. C. Yao, *A viscosity iterative method for a split feasibility problem*, J. Nonlinear Convex Anal. **20** (2019), 1497–1506.

[24] D. R. Sahu, J. C. Yao, M. Verma and K. K. Shukla, *Convergence rate analysis of proximal gradient methods with applications to composite minimization problems*, Optimization **70** (2021), 75–100.

[25] Y. Shehu, Q. L. Dong and D. Jiang, *Single projection method for pseudo-monotone variational inequality in Hilbert spaces*, Optimization **68** (2019), 385–409.

[26] Y. Shehu and A. Gibali, *New inertial relaxed method for solving split feasibilities*, Optimization **15** (2021), 2109–2126.

[27] Y. Shehu and O. S. Iyiola, *Projection methods with alternating inertial steps for variational inequalities: Weak and linear convergence*, Appl. Numer. Math. **157** (2020), 315–337.

[28] Y. Shehu, X. H. Li and Q. L. Dong, *An efficient projection-type method for monotone variational inequalities in Hilbert spaces*, Numer. Algorithms **84** (2020), 365–388.

[29] Y. Shehu, O. S. Iyiola and F. U. Ogbuisi, *Iterative method with inertial terms for nonexpansive mappings: applications to compressed sensing*, Numer. Algo. **83** (2020), 1321–1347.

[30] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.

[31] B. Tan and S. Y. Cho, *Self-adaptive inertial shrinking projection algorithms for solving pseudomonotone variational inequalities*, J. Nonlinear Convex Anal. **22** (2021), 13–27.

[32] B. Tan and S. Y. Cho, *Inertial extragradient methods for solving pseudomonotone variational inequalities with non-Lipschitz mappings and their optimization applications*, Appl. Set-Valued Anal. Optim. **3** (2021), 165–192.

[33] B. Tan, J. Fan and S. Li, *Self-adaptive inertial extragradient algorithms for solving variational inequality problems*, Comput. Appl. Math. **40** (2021), Article ID 19.

[34] D. V. Thong and D.V. Hieu, *Some extragradient-viscosity algorithms for solving variational inequality problems and fixed point problems*, Numer. Algorithms **82** (2019), 761–789.

[35] P. Tseng, *A modified forward-backward splitting method for maximal monotone mappings*, SIAM J. Control Optim. **38** (2000), 431–446.

[36] J. Yang, *Self-adaptive inertial subgradient extragradient algorithm for solving pseudomonotone variational inequalities*, Appl. Anal. **100** (2021), 1067–1078.

[37] J. Yang and H. Liu, *Strong convergence result for solving monotone variational inequalities in Hilbert space*, Numer. Algorithms **80** (2019), 741–752.

[38] Z. Zhou, B. Tan and S. Li, *A new accelerated self-adaptive stepsize algorithm with excellent stability for split common fixed point problems*, Comput. Appl. Math. **39** (2020), Article ID 220.

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