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Inertial extragradient algorithms with non-monotone stepsizes for pseudomonotone variational inequalities and applications

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Abstract
The goal of this paper is to construct several fast iterative algorithms for solving pseudomonotone variational inequalities in real Hilbert spaces. We introduce two extragradient algorithms with inertial terms and give a strong convergence analysis under suitable assumptions. The suggested algorithms need to compute the projection on the feasible set only once in each iteration and can update the step size adaptively without any line search condition. Some numerical experiments and applications are implemented to illustrate the advantages and efficiency of the suggested algorithms over the related known methods.

Keywords Variational inequality problem · Subgradient extragradient method · Tseng's extragradient method · Inertial method · Pseudomonotone mapping

Mathematics Subject Classification 47J20 · 47J25 · 47J30 · 68W10 · 65K15

1 Introduction

The purpose of this paper is to construct several adaptive fast iterative algorithms to solve the following variational inequality problem (shortly, VIP) in real Hilbert spaces:

$$\text{find } x^* \in C \text{ such that } \langle Ax^*, z - x^* \rangle \geq 0, \quad \forall z \in C, \qquad (VIP)$$

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where C is a nonempty, closed, and convex subset in a real Hilbert space $\mathcal{H}$ associated with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$, and $A : \mathcal{H} \to \mathcal{H}$ is a single-valued nonlinear mapping. The solution set of the variational inequality problem (VIP) is denoted by $\mathrm{VI}(C, A)$. For every point $x \in \mathcal{H}$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $P_C(x) := \operatorname{argmin}\{\|x - y\|, y \in C\}$. P_C is called the metric projection of $\mathcal{H}$ onto C. To begin with, let us recall the following concepts in convex and nonlinear analysis. Recall that an operator $A : \mathcal{H} \to \mathcal{H}$ is said to be:

(i) *L-Lipschitz continuous* with $L > 0$ if $\|Ax - Ay\| \leq L\|x - y\|$, $\forall x, y \in \mathcal{H}$ (if $L = 1$, then A is called nonexpansive);
(ii) *β-strongly monotone* if there exists $\beta > 0$ such that $\langle Ax - Ay, x - y \rangle \geq \beta\|x - y\|^2$, $\forall x, y \in \mathcal{H}$;
(iii) *monotone* if $\langle Ax - Ay, x - y \rangle \geq 0$, $\forall x, y \in \mathcal{H}$;
(iv) *pseudomonotone* if $\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, y - x \rangle \geq 0$, $\forall x, y \in \mathcal{H}$;
(v) *sequentially weakly continuous* if for each sequence $\{x_n\}$ converges weakly to x implies $\{Ax_n\}$ converges weakly to Ax.

It is easy to see that the following relation: (ii)$\Rightarrow$(iii)$\Rightarrow$(iv). However, the opposite statements do not hold in general.

Variational inequality problems provide an effective and critical tool for studying many interesting problems that arise in different fields, such as physics, engineering, economics, mathematical programming, and many more (see, e.g., Cuong et al. 2020; Tan et al. 2020; An et al. 2021; Sahu et al. 2021). In the past few decades, many effective numerical methods were developed and investigated to solve variational inequalities and related optimization problems; see, e.g., Thong and Hieu (2018), Gibali and Thong (2020), Shehu et al. (2020), Shehu et al. (2019), Tan et al. (2020) and the references therein. In this paper, we focus on the projection-type methods and their variants. In particular, we recommend the reader to refer to the extragradient method introduced by Korpelevich (Korpelevich 1976), the Tseng's extragradient method (Tseng 2000), and the subgradient extragradient method proposed by Censor et al. (2011). Notice that the extragradient method in Korpelevich (1976) requires computing the projection on the feasible set twice in each iteration, which affects its computational efficiency if the feasible set is complex. In contrast, the two modified extragradient methods proposed in Tseng (2000), Censor et al. (2011) only need to compute the projection on the feasible set once in each iteration, which greatly improves the computational efficiency of the extragradient method.

The extragradient-type methods for solving variational inequality problems introduced in Shehu et al. (2020), Shehu et al. (2019), Korpelevich (1976), Tseng (2000), Censor et al. (2011) all achieve weak convergence in infinite-dimensional Hilbert spaces. Since the class of pseudomonotone operators contains the class of monotone operators and the pseudomonotone operators have a wider application in practice, recently some authors extended the extragradient-type methods to solve pseudomonotone variational inequality problems (see, e.g., Vuong 2018; Shehu et al. 2019). Under some suitable conditions, they obtained weak convergence theorems for the algorithms presented in Vuong (2018), Shehu et al. (2019). It is known that strong convergence is preferable to weak convergence in infinite-dimensional spaces. When mapping A is pseudomonotone and L-Lipschitz continuous but L is unknown, a natural problem is how to modify the extragradient-type algorithm to solve the (VIP) and maintain the strong convergence of the algorithm used. Recently, Thong and Vuong (Thong and Vuong 2019) introduced a modified Mann-type Tseng's extragradient method to address pseudomonotone variational inequality problems in real Hilbert spaces. They used an Armijo line search method to eliminate the dependence on the Lipschitz continuous modulus of the mapping A. Indeed, their algorithm has the following form

$$\begin{cases} y_n = P_C\left(x_n - \tau_n A x_n\right), \\ z_n = y_n - \tau_n\left(A y_n - A x_n\right), \\ x_{n+1} = \left(1 - \alpha_n - \beta_n\right) x_n + \beta_n z_n, \quad \forall n \geq 1, \end{cases} \tag{MaTEGM}$$

where the mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, sequentially weakly continuous on C, and uniformly continuous on bounded subsets of $\mathcal{H}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $(0, 1)$ such that $\{\beta_n\} \subset (a, 1 - \alpha_n)$ for some $a > 0$, $\lim_{n\to\infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. The step size τ_n is updated by the Armijo line search criterion (1.1). Set $\tau_n := \gamma \ell^{m_n}$ and m_n is the smallest non-negative integer m satisfying

$$\gamma \ell^m \|A x_n - A y_n\| \leq \mu \|x_n - y_n\|, \quad \gamma > 0, \ell \in (0, 1), \mu \in (0, 1). \tag{1.1}$$

They proved that the iterative sequence defined by Algorithm (MaTEGM) converges strongly to an element p under the condition that $\mathrm{VI}(C, A)$ is nonempty, where $p = \arg\min\{\|z\| : z \in \mathrm{VI}(C, A)\}$. Notice that the algorithm (MaTEGM) can work adaptively because it uses the Armijo criterion to automatically update the iteration step sizes.

To accelerate the convergence rate of the algorithm used, Polyak (Polyak (1964)) considered the second-order dynamical system $\ddot{x}(t) + \gamma \dot{x}(t) + \nabla f(x(t)) = 0$, where $\gamma > 0$, ∇f is the gradient of f, $\dot{x}(t)$ and $\ddot{x}(t)$ denote the first and second derivatives of x at t, respectively. This dynamic system is called the Heavy Ball with Friction (HBF). Next, consider the discretization of this dynamic system (HBF), that is,

$$\frac{x_{n+1} - 2x_n + x_{n-1}}{h^2} + \gamma \frac{x_n - x_{n-1}}{h} + \nabla f\left(x_n\right) = 0, \quad \forall n \geq 1.$$

Through a direct calculation, we can obtain the following form

$$x_{n+1} = x_n + \beta\left(x_n - x_{n-1}\right) - \alpha \nabla f\left(x_n\right), \quad \forall n \geq 1,$$

where $\beta = 1 - \gamma h$ and $\alpha = h^2$. This can be thought of as the following two-step iterative scheme

$$\begin{cases} y_n = x_n + \beta\left(x_n - x_{n-1}\right), \\ x_{n+1} = y_n - \alpha \nabla f\left(x_n\right), \quad \forall n \geq 1. \end{cases}$$

This iteration is now called the inertial extrapolation algorithm, and the term $\beta\left(x_n - x_{n-1}\right)$ is referred to as the extrapolation point. In recent years, the inertial technique, used as an acceleration method, attracted extensive research and interest by scholars who constructed a large number of fast iterative algorithms to address variational inequalities, split feasibility problems, fixed point problems, and other optimization problems; see, e.g., Gibali and Hieu (2019), Tan et al. (2021), Hieu and Gibali (2020), Shehu and Gibali (2021), Shehu and Iyiola (2020), Sahu et al. (2021), Tan and Li (2020) and the references therein. These algorithms demonstrate advantages in both theory and numerical experiments.

Note that the algorithm (MaTEGM) uses an Armijo line search criterion to eliminate the Lipschitz constant that may be unknown to the operator A. The use of the Armijo criterion may evaluate multiple times the value of the operator A and the projection on the feasible set, which further affects the computational efficiency of such algorithms. Recently, many authors used a new iteration step size rule in their algorithms to address variational inequality problems. For example, Thong et al. (Thong et al. 2020) introduced a viscosity-type inertial subgradient extragradient algorithm for solving pseudomonotone variational inequality problems in real Hilbert spaces. Their algorithm is described as follows

$$\begin{cases} w_n = x_n + \theta_n (x_n - x_{n-1}), \\ y_n = P_C (w_n - \tau_n A w_n), \\ T_n = \{x \in \mathcal{H} : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \leq 0\}, \\ z_n = P_{T_n} (w_n - \tau_n A y_n), \\ x_{n+1} = \alpha_n f(z_n) + (1 - \alpha_n) z_n, \quad \forall n \geq 1. \end{cases} \tag{ViSEGM}$$

where the stepsize τ_n is generated by

$$\tau_{n+1} = \begin{cases} \min \left\{ \dfrac{\mu \|w_n - y_n\|}{\|A w_n - A y_n\|}, \tau_n \right\}, & \text{if } A w_n - A y_n \neq 0; \\ \tau_n, & \text{otherwise.} \end{cases} \tag{1.2}$$

The mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone, L-Lipschitz continuous, and sequentially weakly continuous on $\mathcal{H}$, and the inertial parameter θ_n is updated by the following way:

$$\theta_n = \begin{cases} \min \left\{ \dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta \right\}, & \text{if } x_n \neq x_{n-1}; \\ \theta, & \text{otherwise.} \end{cases}$$

Note that Algorithm (ViSEGM) applies a simple step size without any line search condition, which is obtained at each iteration by a simple computation of previously known information. Therefore, the algorithm (ViSEGM) can work without the prior knowledge of the Lipschitz constant of the mapping A. They established a strong convergence theorem for Algorithm (ViSEGM) under some mild conditions. It should be mentioned that the step size criterion (1.2) used in Algorithm (ViSEGM) generates a non-increasing sequence of steps, which may affect the execution efficiency of this algorithm. To overcome this drawback, a modified version of the step size criterion (1.2), which generates a non-monotonic sequence of step sizes, was recently introduced by Liu and Yang (Liu and Yang 2020).

Our concern now is the following: *How to accelerate the subgradient extragradient method and Tseng's extragradient method for solving the variational inequality problem without requiring the prior information of the Lipschitz constant of the operator and providing strong convergence?*

In this paper, we introduce two new inertial extragradient algorithms with non-monotone step sizes for solving pseudomonotone variational inequality problems in real Hilbert spaces. The suggested algorithms have several advantages over some known results in the literature. More precisely, the contributions of this paper are stated as follows.

(1) Based on the subgradient extragradient method, the Tseng's extragradient method, the Mann-type method, and the inertial method, we present two new iterative schemes that compute the projection on the feasible set only once in each iteration. This improves the extragradient method introduced in Korpelevich (1976) that requires computing the projection on the feasible set twice in each iteration. Moreover, the inertial effects are also embedded in the proposed algorithms to accelerate their convergence.
(2) Our two algorithms interpolate a non-monotonic step size criterion introduced in Liu and Yang (2020), which allows the suggested algorithms to work adaptively without

the prior knowledge of the Lipschitz constant of the mapping. Moreover, this non-monotonic step size criterion does not contain any line search process, which may improve the computational efficiency of the method (MaTEGM) with the Armijo step size criterion (1.1) introduced in Thong and Vuong (2019) and the algorithm (ViSEGM) with the non-increasing step size criterion (1.2) presented in Thong et al. (2020).

(3) The strong convergence of the iterative sequence generated by the proposed algorithms is established in the case that the operator A is pseudomonotone. In other words, our algorithms can solve pseudomonotone variational inequality problems in real Hilbert spaces and can obtain strong convergence, which improves and generalizes many results in the literature (see, e.g., Thong and Hieu 2018; Shehu et al. 2020, 2019; Korpelevich 1976; Tseng 2000; Censor et al. 2011) for solving monotone variational inequality problems. Our algorithms also improve many weakly convergent methods in the literature (see, e.g., (Gibali and Thong 2020; Korpelevich 1976; Tseng 2000; Censor et al. 2011; Vuong 2018)) for solving variational inequalities in real Hilbert spaces.

(4) Some numerical examples occurring in finite- and infinite-dimensional spaces and applications in optimal control problems are given to support the theoretical results of this paper.

The remainder of this paper is organized as follows. In Sect. 2, we recall some preliminary results and lemmas that need to be used in the sequel. Section 3 presents two adaptive inertial extragradient algorithms and analyzes their convergence. Some numerical examples and applications are provided in Sect. 4 to illustrate the numerical behavior of the proposed algorithms and to compare them with some existing ones. Finally, a brief summary of this paper is given in Sect. 5, the last section.

2 Preliminaries

Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$. The weak convergence and strong convergence of $\{x_n\}$ to x are represented by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. For each $x, y, z \in \mathcal{H}$, we have the following inequalities:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.1}$$

and

$$\begin{aligned}\|\alpha x+\beta y+\gamma z\|^2 = {} & \alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2 \\ & -\beta\gamma\|y-z\|^2, \text{ where } \alpha, \beta, \gamma \in [0,1] \text{ with } \alpha+\beta+\gamma=1.\end{aligned} \tag{2.2}$$

It is known that P_C is nonexpansive and it has the following basic properties:

$$\langle x - P_C(x), y - P_C(x)\rangle \leq 0, \quad \forall x \in \mathcal{H}, y \in C, \tag{2.3}$$

and

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y\rangle, \quad \forall x, y \in \mathcal{H}. \tag{2.4}$$

We need the following lemmas to prove the convergence of the suggested algorithms.

Lemma 2.1 (Cottle and Yao 1992) *Let C be a nonempty, closed, and convex subset of a real Hilbert space $\mathcal{H}$ and $A : C \to \mathcal{H}$ be a continuous and pseudomonotone operator. Then, $x^* \in C$ is a solution of* $\mathrm{VI}(C, A)$ *if and only if $\langle Ax, x - x^* \rangle \geq 0,\ \forall x \in C$.*

Lemma 2.2 (Maingé 2008) *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{a_{n_j}\}$ of $\{a_n\}$ such that $a_{n_j} < a_{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty} m_k = \infty$ and the following properties are satisfied by all (sufficiently large) number $k \in \mathbb{N}$:*

$$a_{m_k} \leq a_{m_k+1} \text{ and } a_k \leq a_{m_k+1}.$$

In fact, m_k is the largest number n in the set $\{1, 2, \ldots, k\}$ such that $a_n < a_{n+1}$.

Lemma 2.3 (Xu 2002) *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n) a_n + \alpha_n b_n, \quad \forall n > 1,$$

where $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ is a real sequence that satisfies $\limsup_{n\to\infty} b_n \leq 0$. *Then,* $\lim_{n\to\infty} a_n = 0$.

3 Main results

In this section, we introduce two new inertial extragradient algorithms with a new non-monotonic step size rule for solving pseudomonotone variational inequality problems in real Hilbert spaces and analyze their convergence. First, we assume that the proposed algorithms satisfy the following conditions.

(C1) The feasible set C is a nonempty, closed, and convex subset of $\mathcal{H}$, and the solution set of (VIP) is nonempty, i.e., $\mathrm{VI}(C, A) \neq \emptyset$.
(C2) The mapping $A : \mathcal{H} \to \mathcal{H}$ is pseudomonotone and L-Lipschitz continuous on $\mathcal{H}$, and sequentially weakly continuous on C.
(C3) Let $\{\xi_n\}$ and $\{\epsilon_n\}$ be two positive sequences such that $\sum_{n=1}^{\infty} \xi_n < \infty$ and $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$, where $\{\alpha_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Let $\{\beta_n\} \subset (a, b) \subset (0, 1 - \alpha_n)$ for some $a > 0$ and $b > 0$.

3.1 The Mann-type inertial subgradient extragradient algorithm

Now, we introduce a Mann-type inertial subgradient extragradient algorithm to solve pseudomonotone variational inequality problems. The form of our Algorithm 3.1 is described as follows.

Algorithm 3.1 The Mann-type inertial subgradient extragradient algorithm

Initialization: Take $\theta > 0$, $\tau_1 > 0$, $\mu \in (0,1)$. Choose sequences $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\xi_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$. Calculate x_{n+1} as follows:
Step 1. Compute $w_n = x_n + \theta_n \left(x_n - x_{n-1}\right)$, where

$$\theta_n = \begin{cases} \min\left\{\dfrac{\epsilon_n}{\|x_n - x_{n-1}\|}, \theta\right\}, & \text{if } x_n \neq x_{n-1}; \\ \theta, & \text{otherwise.} \end{cases} \tag{3.1}$$

Step 2. Compute $y_n = P_C(w_n - \tau_n A w_n)$. If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $z_n = P_{T_n}(w_n - \tau_n A y_n)$, where the half space T_n is constructed as follows

$$T_n := \{x \in \mathcal{H} : \langle w_n - \tau_n A w_n - y_n, x - y_n \rangle \leq 0\}.$$

Step 4. Compute $x_{n+1} = (1 - \alpha_n - \beta_n) w_n + \beta_n z_n$, and update the step size τ_{n+1} by

$$\tau_{n+1} = \begin{cases} \min\left\{\dfrac{\mu\|w_n - y_n\|}{\|A w_n - A y_n\|}, \tau_n + \xi_n\right\}, & \text{if } A w_n - A y_n \neq 0; \\ \tau_n + \xi_n, & \text{otherwise.} \end{cases} \tag{3.2}$$

Set $n := n+1$ and go to **Step 1**.

Remark 3.1 It follows from (3.1) that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0.$$

Indeed, we have $\theta_n \|x_n - x_{n-1}\| \leq \epsilon_n$ for all $n \geq 1$, which together with $\lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0$ implies that

$$\lim_{n\to\infty} \frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\alpha_n} = 0.$$

Applying similar statements as in the proof of (Liu and Yang 2020, Lemma 3.1), we can obtain the following Lemma 3.1.

Lemma 3.1 *Suppose that Conditions (C1) and (C2) hold. The sequence* $\{\tau_n\}$ *generated by (3.2) is well defined and* $\lim_{n\to\infty} \tau_n = \tau$ *and* $\tau \in \left[\min\left\{\frac{\mu}{L}, \tau_1\right\}, \tau_1 + \Xi\right]$, *where* $\Xi = \sum_{n=1}^{\infty} \xi_n$.

Proof Since mapping M is L-Lipschitz continuous, one has

$$\frac{\mu\|w_n - y_n\|}{\|A w_n - A y_n\|} \geq \frac{\mu\|w_n - y_n\|}{L\|w_n - y_n\|} = \frac{\mu}{L}, \text{ if } A w_n \neq A y_n.$$

Thus, $\tau_n \geq \min\left\{\frac{\mu}{L}, \tau_1\right\}$. It follows from the definition of τ_{n+1} that $\tau_{n+1} \leq \tau_1 + \Xi$. Consequently, the sequence $\{\tau_n\}$ defined in (3.2) is bounded and $\tau_n \in \left[\min\left\{\frac{\mu}{L}, \tau_1\right\}, \tau_1 + \Xi\right]$. For simplicity, we define $(\tau_{n+1} - \tau_n)^+ = \max\{0, \tau_{n+1} - \tau_n\}$ and $(\tau_{n+1} - \tau_n)^- = \max\{0, -(\tau_{n+1} - \tau_n)\}$. By the definition of $\{\tau_n\}$, one obtains $\sum_{n=1}^{\infty}(\tau_{n+1} - \tau_n)^+ \leq \sum_{n=1}^{\infty} \xi_n < +\infty$, which implies that the series $\sum_{n=1}^{\infty}(\tau_{n+1} - \tau_n)^+$ is convergent. Next we show the convergence of the series $\sum_{n=1}^{\infty}(\tau_{n+1} - \tau_n)^-$. Suppose that $\sum_{n=1}^{\infty}(\tau_{n+1} - \tau_n)^- = +\infty$. Note that $\tau_{n+1} - \tau_n = (\tau_{n+1} - \tau_n)^+ - (\tau_{n+1} - \tau_n)^-$. Therefore,

$$\tau_{k+1} - \tau_1 = \sum_{n=1}^{k}(\tau_{n+1} - \tau_n) = \sum_{n=1}^{k}(\tau_{n+1} - \tau_n)^+ - \sum_{n=1}^{k}(\tau_{n+1} - \tau_n)^-.$$

Taking $k \to +\infty$ in the above equation, we obtain $\lim_{k\to+\infty} \tau_k \to -\infty$. That is a contradiction. Hence, we deduce that $\lim_{n\to\infty} \tau_n = \tau$ and $\tau \in \left[\min\left\{\frac{\mu}{L}, \tau_1\right\}, \tau_1 + \Xi\right]$. □

Remark 3.2 The idea of the step size τ_n defined in (3.2) is derived from Liu and Yang (2020). It is worth noting that the step size τ_n generated in Algorithm 3.1 is allowed to increase from iteration to iteration. Therefore, the use of this type of step size reduces the dependence on the initial step size τ_1. On the other hand, because of $\sum_{n=1}^{\infty} \xi_n < +\infty$, which implies that $\lim_{n\to\infty} \xi_n = 0$. Consequently, the step size τ_n may not increase when n is large enough. If $\xi_n = 0$, then the step size τ_n in Algorithm 3.1 is similar to the approaches in Shehu and Iyiola (2020); Thong et al. (2020).

The following lemmas are quite helpful to analyze the convergence of our main results.

Lemma 3.2 *Assume that Conditions (C1) and (C2) hold. Let $\{z_n\}$ be a sequence created by Algorithm 3.1. Then, for all $p \in \mathrm{VI}(C, A)$,*

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|y_n - w_n\|^2 - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right)\|z_n - y_n\|^2 .$$

Proof Using the definition of $\{\tau_n\}$, one obtains

$$\|Aw_n - Ay_n\| \le \frac{\mu}{\tau_{n+1}}\|w_n - y_n\|, \quad \forall n \ge 1. \tag{3.3}$$

Indeed, if $Aw_n = Ay_n$ then inequality (3.3) holds. Otherwise, it implies from (3.2) that

$$\tau_{n+1} = \min\left\{\frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}, \tau_n + \xi_n\right\} \le \frac{\mu\|w_n - y_n\|}{\|Aw_n - Ay_n\|}.$$

Consequently,

$$\|Aw_n - Ay_n\| \le \frac{\mu}{\tau_{n+1}}\|w_n - y_n\| .$$

Therefore, the inequality (3.3) holds in any case. Note that $p \in T_n$. By the definition of z_n and (2.4), one sees that

$$\begin{aligned}
2\|z_n - p\|^2 &= 2\left\|P_{T_n}(w_n - \tau_n Ay_n) - P_{T_n}(p)\right\|^2 \\
&\le 2\langle z_n - p, w_n - \tau_n Ay_n - p\rangle \\
&= \|z_n - p\|^2 + \|w_n - \tau_n Ay_n - p\|^2 - \|z_n - w_n + \tau_n Ay_n\|^2 \\
&= \|z_n - p\|^2 + \|w_n - p\|^2 + \tau_n^2\|Ay_n\|^2 - 2\langle w_n - p, \tau_n Ay_n\rangle \\
&\quad - \|z_n - w_n\|^2 - \tau_n^2\|Ay_n\|^2 - 2\langle z_n - w_n, \tau_n Ay_n\rangle \\
&= \|z_n - p\|^2 + \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \tau_n Ay_n\rangle .
\end{aligned}$$

This implies that

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \|z_n - w_n\|^2 - 2\langle z_n - p, \tau_n Ay_n\rangle . \tag{3.4}$$

Since p is the solution of (VIP), we have $\langle Ap, x - p\rangle \ge 0$ for all $x \in C$. By the pseudomonotonicity of mapping A, we obtain $\langle Ax, x - p\rangle \ge 0$ for all $x \in C$. Taking $x = y_n \in C$, one infers that $\langle Ay_n, p - y_n\rangle \le 0$. Consequently,

$$\langle Ay_n, p - z_n\rangle = \langle Ay_n, p - y_n\rangle + \langle Ay_n, y_n - z_n\rangle \le \langle Ay_n, y_n - z_n\rangle . \tag{3.5}$$

Combining (3.4) and (3.5), one obtains

$$\begin{aligned}\|z_n - p\|^2 &\le \|w_n - p\|^2 - \|z_n - w_n\|^2 + 2\tau_n \langle Ay_n, y_n - z_n\rangle \\ &= \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 \\ &\quad - 2\langle z_n - y_n, y_n - w_n\rangle + 2\tau_n \langle Ay_n, y_n - z_n\rangle \\ &= \|w_n - p\|^2 - \|z_n - y_n\|^2 - \|y_n - w_n\|^2 \\ &\quad + 2\langle z_n - y_n, w_n - \tau_n Ay_n - y_n\rangle .\end{aligned} \tag{3.6}$$

From $z_n \in T_n$, one has $\langle w_n - \tau_n Aw_n - y_n, z_n - y_n\rangle \le 0$. Thus,

$$\begin{aligned}&2\langle w_n - \tau_n Ay_n - y_n, z_n - y_n\rangle \\ &= 2\langle w_n - \tau_n Aw_n - y_n, z_n - y_n\rangle + 2\tau_n \langle Aw_n - Ay_n, z_n - y_n\rangle \\ &\le 2\tau_n \langle Aw_n - Ay_n, z_n - y_n\rangle .\end{aligned} \tag{3.7}$$

Next, we estimate $2\tau_n \langle Aw_n - Ay_n, z_n - y_n\rangle$. In view of (3.3), we deduce

$$\begin{aligned}&2\tau_n \langle Aw_n - Ay_n, z_n - y_n\rangle \\ &\le 2\tau_n \|Ay_n - Aw_n\| \|y_n - z_n\| \le 2\mu\frac{\tau_n}{\tau_{n+1}} \|w_n - y_n\| \|y_n - z_n\| \\ &\le \mu\frac{\tau_n}{\tau_{n+1}} \|w_n - y_n\|^2 + \mu\frac{\tau_n}{\tau_{n+1}} \|y_n - z_n\|^2 .\end{aligned} \tag{3.8}$$

Combining (3.6), (3.7), and (3.8), we obtain

$$\|z_n - p\|^2 \le \|w_n - p\|^2 - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right) \|y_n - w_n\|^2 - \left(1 - \mu\frac{\tau_n}{\tau_{n+1}}\right) \|z_n - y_n\|^2 .$$

This completes the proof. □

Lemma 3.3 *Assume that Conditions (C1) and (C2) hold. Let $\{w_n\}$ be a sequence formed by Algorithm 3.1. If there exists a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ that converges weakly to $z \in \mathcal{H}$ and $\lim_{k\to\infty} \|w_{n_k} - y_{n_k}\| = 0$, then $z \in \mathrm{VI}(C, A)$.*

Proof From $y_n = P_C (w_n - \tau_n Aw_n)$ and (2.3), we have

$$\langle w_{n_k} - \tau_{n_k} Aw_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le 0, \quad \forall x \in C,$$

or equivalently

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle \le \langle Aw_{n_k}, x - y_{n_k}\rangle, \quad \forall x \in C.$$

This implies that

$$\frac{1}{\tau_{n_k}} \langle w_{n_k} - y_{n_k}, x - y_{n_k}\rangle + \langle Aw_{n_k}, y_{n_k} - w_{n_k}\rangle \le \langle Aw_{n_k}, x - w_{n_k}\rangle, \quad \forall x \in C. \tag{3.9}$$

We have $\{w_{n_k}\}$ is bounded since $\{w_{n_k}\}$ converges weakly to $z \in \mathcal{H}$. From the Lipschitz continuity of mapping A and $\|w_{n_k} - y_{n_k}\| \to 0$, we obtain $\{Aw_{n_k}\}$ and $\{y_{n_k}\}$ are also bounded. Since $\tau_{n_k} \ge \min\{\tau_1, \frac{\mu}{L}\}$, one concludes from (3.9) that

$$\liminf_{k\to\infty} \langle Aw_{n_k}, x - w_{n_k}\rangle \ge 0, \quad \forall x \in C. \tag{3.10}$$

Moreover, one sees that

$$\langle Ay_{n_k}, x - y_{n_k}\rangle = \langle Ay_{n_k} - Aw_{n_k}, x - w_{n_k}\rangle + \langle Aw_{n_k}, x - w_{n_k}\rangle + \langle Ay_{n_k}, w_{n_k} - y_{n_k}\rangle . \tag{3.11}$$

According to the fact that A is Lipschitz continuous and $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0$, it follows that $\lim_{k\to\infty}\|Aw_{n_k}-Ay_{n_k}\|=0$. This together with (3.10) and (3.11) yields that

$$\liminf_{k\to\infty}\left\langle Ay_{n_k}, x-y_{n_k}\right\rangle \geq 0.$$

Next, we show that $z\in \mathrm{VI}(C,A)$. We choose a positive numbers decreasing sequence $\{\zeta_k\}$ such that $\zeta_k\to 0$ as $k\to\infty$. For any k, we denote by N_k the smallest positive integer such that

$$\left\langle Ay_{n_j}, x-y_{n_j}\right\rangle + \zeta_k \geq 0, \quad \forall j\geq N_k. \tag{3.12}$$

It is easy to see that the sequence $\{N_k\}$ is increasing because of the sequence $\{\zeta_k\}$ is decreasing. Furthermore, for any k, since $\{y_{N_k}\}\subset C$ we can assume $Ay_{N_k}\neq 0$ (otherwise, y_{N_k} is a solution) and set $u_{N_k}=Ay_{N_k}/\|Ay_{N_k}\|^2$. Then, we obtain $\left\langle Ay_{N_k}, u_{N_k}\right\rangle = 1$ for all $k\geq 1$. It follows from (3.12) that

$$\left\langle Ay_{N_k}, x+\zeta_k u_{N_k}-y_{N_k}\right\rangle \geq 0, \quad \forall k\geq 1.$$

By the fact that A is pseudomonotone on $\mathcal{H}$, we have

$$\left\langle A\left(x+\zeta_k u_{N_k}\right), x+\zeta_k u_{N_k}-y_{N_k}\right\rangle \geq 0.$$

This implies that

$$\left\langle Ax, x-y_{N_k}\right\rangle \geq \left\langle Ax - A\left(x+\zeta_k u_{N_k}\right), x+\zeta_k u_{N_k}-y_{N_k}\right\rangle - \zeta_k\left\langle Ax, u_{N_k}\right\rangle. \tag{3.13}$$

Now, we show that $\lim_{k\to\infty}\zeta_k u_{N_k}=0$. Indeed, we obtain $y_{N_k}\rightharpoonup z$ since $w_{n_k}\rightharpoonup z$ and $\lim_{k\to\infty}\|w_{n_k}-y_{n_k}\|=0$. From $\{y_n\}\subset C$, we have $z\in C$. Since A is sequentially weakly continuous on C, $\{Ay_{n_k}\}$ converges weakly to Az. We can assume $Az\neq 0$ (otherwise, z is a solution). Using the fact that the norm mapping is sequentially weakly lower semicontinuous, we obtain $0<\|Az\|\leq \liminf_{k\to\infty}\|Ay_{n_k}\|$. Since $\{y_{N_k}\}\subset\{y_{n_k}\}$ and $\zeta_k\to 0$ as $k\to\infty$, we have

$$0\leq \limsup_{k\to\infty}\|\zeta_k u_{N_k}\| = \limsup_{k\to\infty}\left(\frac{\zeta_k}{\|Ay_{n_k}\|}\right) \leq \frac{\limsup_{k\to\infty}\zeta_k}{\liminf_{k\to\infty}\|Ay_{n_k}\|} = 0.$$

That is, $\lim_{k\to\infty}\zeta_k u_{N_k}=0$. Combining the Lipschitz continuity of mapping A, $\{y_{N_k}\}$ and $\{u_{N_k}\}$ are bounded, and $\lim_{k\to\infty}\zeta_k u_{N_k}=0$, we can conclude from (3.13) that

$$\liminf_{k\to\infty}\left\langle Ax, x-y_{N_k}\right\rangle \geq 0.$$

Consequently, we have, for all $x\in C$,

$$\langle Ax, x-z\rangle = \lim_{k\to\infty}\left\langle Ax, x-y_{N_k}\right\rangle = \liminf_{k\to\infty}\left\langle Ax, x-y_{N_k}\right\rangle \geq 0.$$

Thus we observe that $z\in\mathrm{VI}(C,A)$ by means of Lemma 2.1. This completes the proof. □

Remark 3.3 It is not necessary to impose the sequential weak continuity of mapping A if A is monotone (see (Denisov et al. 2015)).

Theorem 3.1 *Assume that Conditions (C1)–(C3) hold. Then the sequence* $\{x_n\}$ *generated by Algorithm 3.1 converges to* $p\in\mathrm{VI}(C,A)$ *in norm, where* $\|p\|=\min\{\|z\|: z\in\mathrm{VI}(C,A)\}$.

Proof According to Lemma 3.1, it follows that $\lim_{n\to\infty}\left(1-\mu\frac{\tau_n}{\tau_{n+1}}\right)=1-\mu>0$. Hence, there exists $n_0\in\mathbb{N}$ such that

$$1-\mu\frac{\tau_n}{\tau_{n+1}}>0,\quad \forall n\geq n_0. \tag{3.14}$$

Combining Lemma 3.2 and (3.14), we obtain

$$\|z_n-p\|\leq\|w_n-p\|,\quad \forall n\geq n_0. \tag{3.15}$$

We divided the proof into four claims.

Claim 1. The sequences $\{x_n\}$, $\{w_n\}$, and $\{z_n\}$ are bounded. By the definition of x_{n+1}, one has

$$\begin{aligned}\|x_{n+1}-p\|&=\|(1-\alpha_n-\beta_n)\,w_n+\beta_n z_n-p\|\\&=\|(1-\alpha_n-\beta_n)\,(w_n-p)+\beta_n\,(z_n-p)-\alpha_n p\|\\&\leq\|(1-\alpha_n-\beta_n)\,(w_n-p)+\beta_n\,(z_n-p)\|+\alpha_n\|p\|.\end{aligned} \tag{3.16}$$

It follows from (3.15) that

$$\begin{aligned}&\|(1-\alpha_n-\beta_n)\,(w_n-p)+\beta_n\,(z_n-p)\|^2\\&=(1-\alpha_n-\beta_n)^2\,\|w_n-p\|^2+2\,(1-\alpha_n-\beta_n)\,\beta_n\,\langle w_n-p,z_n-p\rangle+\beta_n^2\,\|z_n-p\|^2\\&\leq(1-\alpha_n-\beta_n)^2\,\|w_n-p\|^2+2\,(1-\alpha_n-\beta_n)\,\beta_n\,\|z_n-p\|\,\|w_n-p\|+\beta_n^2\,\|z_n-p\|^2\\&\leq(1-\alpha_n-\beta_n)^2\,\|w_n-p\|^2+2\,(1-\alpha_n-\beta_n)\,\beta_n\,\|w_n-p\|^2+\beta_n^2\,\|w_n-p\|^2\\&=(1-\alpha_n)^2\,\|w_n-p\|^2,\quad\forall n\geq n_0,\end{aligned}$$

which yields

$$\|(1-\alpha_n-\beta_n)\,(w_n-p)+\beta_n\,(z_n-p)\|\leq(1-\alpha_n)\,\|w_n-p\|,\ \forall n\geq n_0. \tag{3.17}$$

Using the definition of w_n, we can write

$$\begin{aligned}\|w_n-p\|&=\|x_n+\theta_n\,(x_n-x_{n-1})-p\|\\&\leq\|x_n-p\|+\alpha_n\cdot\frac{\theta_n}{\alpha_n}\,\|x_n-x_{n-1}\|.\end{aligned} \tag{3.18}$$

By Remark 3.1, we have $\frac{\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\to 0$. Thus, there exists a constant $M_1>0$ such that

$$\frac{\theta_n}{\alpha_n}\,\|x_n-x_{n-1}\|\leq M_1,\quad\forall n\geq 1. \tag{3.19}$$

From (3.15), (3.18) and (3.19), we find that

$$\|z_n-p\|\leq\|w_n-p\|\leq\|x_n-p\|+\alpha_n M_1,\quad\forall n\geq n_0. \tag{3.20}$$

Combining (3.16), (3.17) and (3.20), we deduce that

$$\begin{aligned}\|x_{n+1}-p\|&\leq(1-\alpha_n)\,\|w_n-p\|+\alpha_n\|p\|\\&\leq(1-\alpha_n)\,\|x_n-p\|+\alpha_n(\|p\|+M_1)\\&\leq\max\{\|x_n-p\|,\|p\|+M_1\},\quad\forall n\geq n_0\\&\leq\cdots\leq\max\left\{\|x_{n_0}-p\|,\|p\|+M_1\right\}.\end{aligned}$$

That is, the sequence $\{x_n\}$ is bounded. By combining $\alpha_n \subset (0,1)$, M_1 is a bounded constant, the boundedness of $\{x_n\}$, and relation (3.20), we can obtain that the sequences $\{z_n\}$ and $\{w_n\}$ are also bounded.

Claim 2.

$$\begin{aligned}&\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|w_n-y_n\|^2+\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_n-z_n\|^2\\&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(\|p\|^2+M_2)\end{aligned}$$

for some $M_2>0$. Using the definition of x_{n+1} and (2.2), one obtains

$$\begin{aligned}\|x_{n+1}-p\|^2&=\|(1-\alpha_n-\beta_n)\,w_n+\beta_n z_n-p\|^2\\&=\|(1-\alpha_n-\beta_n)\,(w_n-p)+\beta_n\,(z_n-p)+\alpha_n(-p)\|^2\\&=(1-\alpha_n-\beta_n)\,\|w_n-p\|^2+\beta_n\,\|z_n-p\|^2+\alpha_n\|p\|^2-\alpha_n\beta_n\,\|z_n\|^2\\&\quad-\beta_n\,(1-\alpha_n-\beta_n)\,\|w_n-z_n\|^2-\alpha_n\,(1-\alpha_n-\beta_n)\,\|w_n\|^2\\&\le(1-\alpha_n-\beta_n)\,\|w_n-p\|^2+\beta_n\,\|z_n-p\|^2+\alpha_n\|p\|^2.\end{aligned}\tag{3.21}$$

In view of (3.20), one sees that

$$\begin{aligned}\|w_n-p\|^2&\le(\|x_n-p\|+\alpha_n M_1)^2\\&=\|x_n-p\|^2+\alpha_n\left(2M_1\,\|x_n-p\|+\alpha_n M_1^2\right)\\&\le\|x_n-p\|^2+\alpha_n M_2\end{aligned}\tag{3.22}$$

for some $M_2>0$. Combining Lemma 3.2, (3.21) and (3.22), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2&\le(1-\alpha_n-\beta_n)\,\|w_n-p\|^2+\beta_n\,\|w_n-p\|^2-\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|w_n-y_n\|^2\\&\quad-\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_n-z_n\|^2+\alpha_n\|p\|^2\\&\le\|x_n-p\|^2-\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|w_n-y_n\|^2\\&\quad-\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_n-z_n\|^2+\alpha_n(\|p\|^2+M_2).\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|w_n-y_n\|^2+\beta_n\Big(1-\mu\frac{\tau_n}{\tau_{n+1}}\Big)\|y_n-z_n\|^2\\&\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_n(\|p\|^2+M_2).\end{aligned}$$

Claim 3.

$$\begin{aligned}\|x_{n+1}-p\|^2\le(1-\alpha_n)\,\|x_n-p\|^2+\alpha_n\Big[&2\beta_n\,\|w_n-z_n\|\,\|x_{n+1}-p\|\\&+2\,\langle p,p-x_{n+1}\rangle+\frac{3M\theta_n}{\alpha_n}\,\|x_n-x_{n-1}\|\Big],\ \forall n\ge n_0\end{aligned}$$

for some $M>0$. Indeed, by the definition of w_n, one obtains

$$\begin{aligned}\|w_n-p\|^2&=\|x_n+\theta_n\,(x_n-x_{n-1})-p\|^2\\&=\|x_n-p\|^2+2\theta_n\,\langle x_n-p,x_n-x_{n-1}\rangle+\theta_n^2\,\|x_n-x_{n-1}\|^2\\&\le\|x_n-p\|^2+3M\theta_n\,\|x_n-x_{n-1}\|,\end{aligned}\tag{3.23}$$

where $M := \sup_{n\in\mathbb{N}}\{\|x_n - p\|, \theta\|x_n - x_{n-1}\|\} > 0$. Setting $t_n = (1-\beta_n)w_n + \beta_n z_n$, one has

$$\|t_n - w_n\| = \beta_n\|w_n - z_n\|. \tag{3.24}$$

It follows from (3.20) that

$$\begin{aligned}\|t_n - p\| &= \|(1-\beta_n)(w_n - p) + \beta_n(z_n - p)\|\\ &\le (1-\beta_n)\|w_n - p\| + \beta_n\|w_n - p\|\\ &= \|w_n - p\|, \quad \forall n \ge n_0.\end{aligned} \tag{3.25}$$

From (2.1), (3.23), (3.24), and (3.25), we obtain

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|(1-\beta_n)w_n + \beta_n z_n - \alpha_n w_n - p\|^2\\ &= \|(1-\alpha_n)(t_n - p) - \alpha_n(w_n - t_n) - \alpha_n p\|^2\\ &\le (1-\alpha_n)^2\|t_n - p\|^2 - 2\alpha_n\langle w_n - t_n + p, x_{n+1} - p\rangle\\ &= (1-\alpha_n)^2\|t_n - p\|^2 + 2\alpha_n\langle w_n - t_n, p - x_{n+1}\rangle + 2\alpha_n\langle p, p - x_{n+1}\rangle\\ &\le (1-\alpha_n)\|t_n - p\|^2 + 2\alpha_n\|w_n - t_n\|\|x_{n+1} - p\| + 2\alpha_n\langle p, p - x_{n+1}\rangle\\ &\le (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\Big[2\beta_n\|w_n - z_n\|\|x_{n+1} - p\|\\ &\quad + 2\langle p, p - x_{n+1}\rangle + \frac{3M\theta_n}{\alpha_n}\|x_n - x_{n-1}\|\Big], \forall n \ge n_0.\end{aligned}$$

Claim 4. We prove that the sequence $\{\|x_n - p\|^2\}$ converges to zero by considering two possible cases on the sequence $\{\|x_n - p\|^2\}$.

Case 1. There exists an $N \in \mathbb{N}$, such that $\|x_{n+1} - p\|^2 \le \|x_n - p\|^2$ for all $n \ge N$. This implies that $\lim_{n\to\infty}\|x_n - p\|^2$ exists. Using $\lim_{n\to\infty}\left(1 - \mu\frac{\tau_n}{\tau_n+1}\right) = 1 - \mu > 0$ and Condition (C3), it implies from Claim 2 that

$$\lim_{n\to\infty}\|w_n - y_n\| = 0 \text{ and } \lim_{n\to\infty}\|y_n - z_n\| = 0.$$

This further yields $\lim_{n\to\infty}\|z_n - w_n\| = 0$, which together with the boundedness of $\{x_n\}$ concludes that

$$\lim_{n\to\infty}\beta_n\|w_n - z_n\|\|x_{n+1} - p\| = 0.$$

According to the definition of w_n and Remark 3.1, one obtains

$$\|x_n - w_n\| = \theta_n\|x_n - x_{n-1}\| = \alpha_n\cdot\frac{\theta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0 \text{ as } n \to \infty.$$

On the other hand, one sees that

$$\|x_{n+1} - w_n\| \le \alpha_n\|w_n\| + \beta_n\|z_n - w_n\| \to 0 \text{ as } n \to \infty.$$

This together with $\lim_{n\to\infty}\|x_n - w_n\| = 0$ implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$, such that $x_{n_j} \rightharpoonup q$ and

$$\limsup_{n\to\infty}\langle p, p - x_n\rangle = \lim_{j\to\infty}\langle p, p - x_{n_j}\rangle = \langle p, p - q\rangle.$$

It follows from Lemma 3.1 that $\lim_{n\to\infty}\tau_n=\tau>0$. We obtain $w_{n_j}\rightharpoonup q$ since $\|x_n-w_n\|\to 0$, which together with $\lim_{n\to\infty}\tau_n=\tau>0$ and $\|w_n-y_n\|\to 0$, in the light of Lemma 3.3, yields that $q\in \mathrm{VI}(C,A)$. Since $q\in \mathrm{VI}(C,A)$ and $\|p\|=\min\{\|z\|:z\in \mathrm{VI}(C,A)\}$, that is $p=P_{\mathrm{VI}(C,A)}0$, we deduce that

$$\limsup_{n\to\infty}\langle p,p-x_n\rangle=\langle p,p-q\rangle\le 0.$$

From $\|x_{n+1}-x_n\|\to 0$, we obtain

$$\limsup_{n\to\infty}\langle p,p-x_{n+1}\rangle\le 0.$$

Therefore, using Claim 3 and Remark 3.1 in Lemma 2.3, we conclude that $x_n\to p$.

Case 2. There exists a subsequence $\{\|x_{n_j}-p\|^2\}$ of $\{\|x_n-p\|^2\}$ such that

$$\|x_{n_j}-p\|^2<\|x_{n_j+1}-p\|^2,\quad \forall j\in\mathbb{N}.$$

In this case, it follows from Lemma 2.2 that there exists a nondecreasing sequence $\{m_k\}$ of $\mathbb{N}$ such that $\lim_{k\to\infty}m_k=\infty$ and the following inequalities hold for all $k\in\mathbb{N}$:

$$\left\|x_{m_k}-p\right\|^2\le\|x_{m_k+1}-p\|^2 \text{ and } \|x_k-p\|^2\le\|x_{m_k+1}-p\|^2.$$

By Claim 2, we have

$$\begin{aligned}&\beta_{m_k}\left(1-\mu\frac{\tau_{m_k}}{\tau_{m_k+1}}\right)\left\|w_{m_k}-y_{m_k}\right\|^2+\beta_{m_k}\left(1-\mu\frac{\tau_{m_k}}{\tau_{m_k+1}}\right)\left\|y_{m_k}-z_{m_k}\right\|^2\\&\le\|x_{m_k}-p\|^2-\|x_{m_k+1}-p\|^2+\alpha_{m_k}(\|p\|^2+M_2)\\&\le\alpha_{m_k}(\|p\|^2+M_2).\end{aligned}$$

By means of Condition (C3), we deduce

$$\lim_{k\to\infty}\left\|w_{m_k}-y_{m_k}\right\|=0 \text{ and } \lim_{k\to\infty}\left\|y_{m_k}-z_{m_k}\right\|=0.$$

As proved in the first case, we obtain $\|x_{m_k+1}-x_{m_k}\|\to 0$ and $\limsup_{k\to\infty}\langle p,p-x_{m_k+1}\rangle\le 0$. From Claim 3 and $\|x_{m_k}-p\|^2\le\|x_{m_k+1}-p\|^2$, we have

$$\begin{aligned}\|x_{m_k+1}-p\|^2\le\left(1-\alpha_{m_k}\right)\|x_{m_k+1}-p\|^2+\alpha_{m_k}\Big[&2\beta_{m_k}\left\|w_{m_k}-z_{m_k}\right\|\|x_{m_k+1}-p\|\\&+2\langle p,p-x_{m_k+1}\rangle+\frac{3M\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k}-x_{m_k-1}\|\Big].\end{aligned}$$

This implies that

$$\|x_k-p\|^2\le 2\beta_{m_k}\left\|w_{m_k}-z_{m_k}\right\|\|x_{m_k+1}-p\|+2\left\langle p,p-x_{m_k+1}\right\rangle+\frac{3M\theta_{m_k}}{\alpha_{m_k}}\|x_{m_k}-x_{m_k-1}\|.$$

Therefore, we obtain $\limsup_{k\to\infty}\|x_k-p\|\le 0$, that is, $x_k\to p$. The proof is completed. □

3.2 The Mann-type inertial Tseng's extragradient algorithm

In this subsection, we present a Mann-type inertial Tseng's extragradient algorithm that contains only one projection step in each iteration for solving pseudomonotone variational inequalities in real Hilbert spaces. The second iterative scheme proposed in this paper is shown in Algorithm 3.2 below.

Algorithm 3.2 The Mann-type inertial Tseng's extragradient algorithm

Initialization: Take $\theta > 0$, $\tau_1 > 0$, $\mu \in (0, 1)$. Choose sequences $\{\epsilon_n\}$, $\{\alpha_n\}$, and $\{\xi_n\}$ to satisfy Condition (C3). Let $x_0, x_1 \in \mathcal{H}$ be arbitrary.
Iterative Steps: Given the iterates x_{n-1} and x_n $(n \geq 1)$. Calculate x_{n+1} as follows:
Step 1. Compute $w_n = x_n + \theta_n \left(x_n - x_{n-1}\right)$, where the inertial parameter θ_n is updated by (3.1).
Step 2. Compute $y_n = P_C\left(w_n - \tau_n A w_n\right)$. If $w_n = y_n$, then stop and y_n is a solution of (VIP). Otherwise, go to **Step 3**.
Step 3. Compute $z_n = y_n - \tau_n\left(A y_n - A w_n\right)$.
Step 4. Compute $x_{n+1} = \left(1 - \alpha_n - \beta_n\right) w_n + \beta_n z_n$, and update the step size τ_{n+1} through (3.2).
Set $n := n + 1$ and go to **Step 1**.

Lemma 3.4 *Assume that Conditions (C1) and (C2) hold. Let $\{z_n\}$ be a sequence formed by Algorithm 3.2. Then,*

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \|w_n - y_n\|^2, \quad \forall p \in \mathrm{VI}(C, A)$$

and

$$\|z_n - y_n\| \leq \mu \frac{\tau_n}{\tau_{n+1}} \|w_n - y_n\| .$$

Proof It follows from (3.2) that

$$\|A w_n - A y_n\| \leq \frac{\mu}{\tau_{n+1}} \|w_n - y_n\|, \quad \forall n \geq 1. \tag{3.26}$$

By the definition of z_n, one sees that

$$\begin{aligned}
\|z_n - p\|^2 &= \|y_n - \tau_n\left(A y_n - A w_n\right) - p\|^2 \\
&= \|y_n - p\|^2 + \tau_n^2 \|A y_n - A w_n\|^2 - 2\tau_n \langle y_n - p, A y_n - A w_n\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 + 2\langle y_n - w_n, w_n - p\rangle \\
&\quad + \tau_n^2 \|A y_n - A w_n\|^2 - 2\tau_n \langle y_n - p, A y_n - A w_n\rangle \\
&= \|w_n - p\|^2 + \|y_n - w_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle + 2\langle y_n - w_n, y_n - p\rangle \\
&\quad + \tau_n^2 \|A y_n - A w_n\|^2 - 2\tau_n \langle y_n - p, A y_n - A w_n\rangle \\
&= \|w_n - p\|^2 - \|y_n - w_n\|^2 + 2\langle y_n - w_n, y_n - p\rangle \\
&\quad + \tau_n^2 \|A y_n - A w_n\|^2 - 2\tau_n \langle y_n - p, A y_n - A w_n\rangle .
\end{aligned} \tag{3.27}$$

Using $y_n = P_C\left(w_n - \tau_n A w_n\right)$ and (2.3), we obtain

$$\langle y_n - w_n + \tau_n A w_n, y_n - p\rangle \leq 0,$$

or equivalently

$$\langle y_n - w_n, y_n - p\rangle \leq -\tau_n \langle A w_n, y_n - p\rangle . \tag{3.28}$$

Combining (3.26), (3.27) and (3.28), we have

$$\begin{aligned} \|z_n - p\|^2 &\leq \|w_n - p\|^2 - \|y_n - w_n\|^2 - 2\tau_n \langle Aw_n, y_n - p\rangle \\ &\quad + \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2} \|w_n - y_n\|^2 - 2\tau_n \langle y_n - p, Ay_n - Aw_n\rangle \\ &\leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \|w_n - y_n\|^2 - 2\tau_n \langle y_n - p, Ay_n\rangle . \end{aligned} \tag{3.29}$$

Since $p \in \mathrm{VI}(C, A)$ and $y_n \in C$, we deduce that $\langle Ap, y_n - p\rangle \geq 0$. This together with the pseudomonotonicity of mapping A, we arrive at

$$\langle Ay_n, y_n - p\rangle \geq 0. \tag{3.30}$$

Combining (3.29) and (3.30), we obtain

$$\|z_n - p\|^2 \leq \|w_n - p\|^2 - \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \|w_n - y_n\|^2 .$$

From the definition of z_n and (3.3), we have

$$\|z_n - y_n\| \leq \mu \frac{\tau_n}{\tau_{n+1}} \|w_n - y_n\| .$$

This completes the proof. □

Theorem 3.2 *Assume that Conditions (C1)–(C3) hold. Then the sequence $\{x_n\}$ generated by Algorithm 3.2 converges to $p \in \mathrm{VI}(C, A)$ in norm, where $\|p\| = \min\{\|z\| : z \in \mathrm{VI}(C, A)\}$.*

Proof The proof is similar to that of Theorem 3.1. We omit some details of the proof to avoid redundancy. Since $\lim_{n\to\infty} \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) = 1 - \mu^2 > 0$, there exists $n_1 \in \mathbb{N}$ such that

$$1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2} > 0, \quad \forall n \geq n_1. \tag{3.31}$$

Combining Lemma 3.4 and (3.31), we deduce

$$\|z_n - p\| \leq \|w_n - p\| , \quad \forall n \geq n_1. \tag{3.32}$$

Claim 1. The sequences $\{x_n\}$, $\{w_n\}$, and $\{z_n\}$ are bounded. This conclusion can be obtained by applying the same statements as stated in Claim 1 of Theorem 3.1.
Claim 2.

$$\begin{aligned} &\beta_n \left(1 - \mu^2 \frac{\tau_n^2}{\tau_{n+1}^2}\right) \|w_n - y_n\|^2 + \beta_n (1 - \alpha_n - \beta_n) \|w_n - z_n\|^2 \\ &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \alpha_n (\|p\|^2 + M_2). \end{aligned}$$

From the definition of x_{n+1} and (2.2), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|(1 - \alpha_n - \beta_n) w_n + \beta_n z_n - p\|^2 \\ &\leq (1 - \alpha_n - \beta_n) \|w_n - p\|^2 + \beta_n \|z_n - p\|^2 + \alpha_n \|p\|^2 \\ &\quad - \beta_n (1 - \alpha_n - \beta_n) \|w_n - z_n\|^2 . \end{aligned} \tag{3.33}$$

Combining Lemma 3.4, (3.22) and (3.33), we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \le & (1-\alpha_n-\beta_n)\|w_n-p\|^2+\beta_n\|w_n-p\|^2-\beta_n\Big(1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\Big)\|w_n-y_n\|^2\\
& +\alpha_n\|p\|^2-\beta_n(1-\alpha_n-\beta_n)\|w_n-z_n\|^2\\
\le & \|x_n-p\|^2-\beta_n\Big(1-\mu^2\frac{\tau_n^2}{\tau_{n+1}^2}\Big)\|w_n-y_n\|^2+\alpha_n(\|p\|^2+M_2)\\
& -\beta_n(1-\alpha_n-\beta_n)\|w_n-z_n\|^2 .
\end{aligned}
$$

The desired result can be obtained by simple deformation.
Claim 3.

$$
\begin{aligned}
\|x_{n+1}-p\|^2 \le (1-\alpha_n)\|x_n-p\|^2+\alpha_n\Big[&2\beta_n\|w_n-z_n\|\|x_{n+1}-p\|\\
&+2\langle p, p-x_{n+1}\rangle+\frac{3M\theta_n}{\alpha_n}\|x_n-x_{n-1}\|\Big],\ \forall n\ge n_1.
\end{aligned}
$$

The desired result can be achieved using the same arguments as in the Claim 3 of Theorem 3.1.
Claim 4. The sequence $\{\|x_n-p\|^2\}$ converges to zero. The proof is similar to the Claim 4 of Theorem 3.1. We leave it for the reader to verify. □

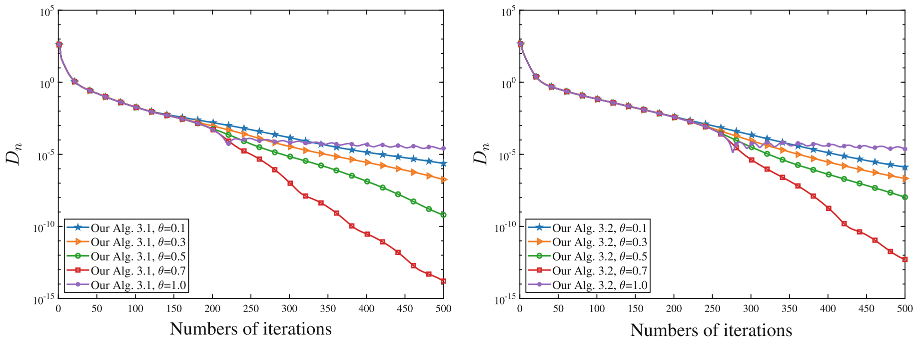
Remark 3.4 If the inertial parameter $\theta_n = 0$ in the proposed Algorithms 3.1 and 3.2, we can obtain two new Mann-type inertial extragradient algorithms with non-monotone step sizes to address the pseudomonotone (VIP) in real Hilbert spaces. These results improve and summarize some recent algorithms in the literature (see, e.g., Thong and Vuong (2019); Hieu et al. (2020); Thong and Hieu (2019); Thong et al. (2019)).

4 Numerical examples and applications

In this section, we provide some numerical examples to show the behavior of the proposed Algorithm 3.1 (shortly, MiSEGM) and Algorithm 3.2 (shortly, MiTEGM), and also to compare them with the following strongly convergent algorithms: the modified Mann-type Tseng's extragradient method (MaTEGM) introduced in Thong and Vuong (2019) and the viscosity-type inertial subgradient extragradient algorithm (ViSEGM) presented in Thong et al. (2020). It is worth noting that these algorithms do not require the prior knowledge of the Lipschitz constant of the operator. All the programs are implemented in MATLAB 2018a on a personal computer.

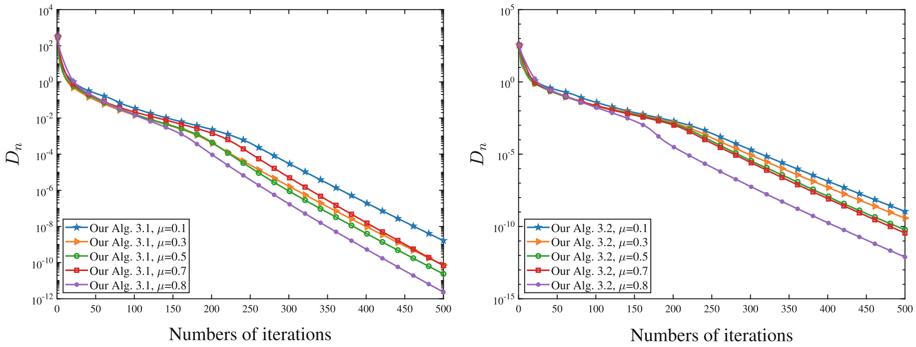
4.1 Theoretical examples

Example 4.1 This example is taken from Harker and Pang (1990) and has been considered by many authors for numerical experiments (see, e.g., Thong and Hieu (2018); Gibali and Thong (2020); Shehu et al. (2020, 2019)). Choose a linear operator $A:\mathbb{R}^m\to\mathbb{R}^m$ as follows $A(x)=Mx+q$, where $q\in\mathbb{R}^m$ and $M=NN^{\mathsf{T}}+U+D$, and N is a $m\times m$ matrix, U is a $m\times m$ skew-symmetric matrix, and D is a $m\times m$ diagonal matrix with its diagonal entries being nonnegative (hence M is positive symmetric definite). The feasible set C is given by $C=\{x\in\mathbb{R}^m : -2\le x_i\le 5,\ i=1,\ldots,m\}$. It is clear that A is monotone and Lipschitz continuous with constant $L=\|M\|$. In this experiment, all entries of N, U are generated



(a) Our Algorithm 3.1 **(b)** Our Algorithm 3.2

Fig. 1 The behavior of our algorithms with different θ in Example 4.1 ($m = 20$)



(a) Our Algorithm 3.1 **(b)** Our Algorithm 3.2

Fig. 2 The behavior of our algorithms with different μ in Example 4.1 ($m = 20$)

randomly and uniformly in $[-2, 2]$, D is generated randomly in $[0, 2]$ and $q = \mathbf{0}$. It is easy to see that the solution of the problem (VIP) in this case is $x^* = \{\mathbf{0}\}$. The initial values $x_0 = x_1$ are randomly generated by *rand(m,1)* in MATLAB. We use $D_n = \|x_n - x^*\|$ to measure the error of the nth iteration step and use the maximum number of iterations 500 as a common stopping criterion for all algorithms. Next, we test the performance of the proposed methods under different parameters. Specifically, we consider the following cases.

Case 1: Compare the inertial parameter θ. Take $\mu = 0.8$, $\tau_1 = 1$, $\theta = \{0.1, 0.3, 0.5, 0.7, 1.0\}$, $\epsilon_n = 1/(n+1)^2$, $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$ and $\xi_n = 1/(n+1)^{1.1}$ for the suggested Algorithms 3.1 and 3.2. The numerical performance of our methods with different parameters θ is stated in Fig. 1.

Case 2: Compare the parameter μ. Take $\mu = \{0.1, 0.3, 0.5, 0.7, 0.8\}$, $\tau_1 = 1$, $\theta = 0.7$, $\epsilon_n = 1/(n+1)^2$, $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$ and $\xi_n = 1/(n+1)^{1.1}$ for the proposed Algorithms 3.1 and 3.2. The numerical behavior of our methods with different parameters θ is given in Fig. 2.

Case 3: Compare the parameter ξ_n. Take $\mu = 0.8$, $\tau_1 = 1$, $\theta = 0.7$, $\epsilon_n = 1/(n+1)^2$, $\alpha_n = 1/(n+1)$ and $\beta_n = 0.9(1-\alpha_n)$ for the proposed Algorithms 3.1 and 3.2. The numerical results of our methods with different parameters ξ_n are reported in Table 1.

Table 1 Numerical results of our algorithms with different ξ_n in Example 4.1 ($m = 20$)

Algorithms	$\xi_n = 0$		$\xi_n = 1/(n+1)^{1.1}$		$\xi_n = 1/(n+1)^3$	
	D_n	CPU (s)	D_n	CPU (s)	D_n	CPU (s)
Our Alg. 3.1	4.30E-07	0.0434	2.72E-09	0.0340	5.97E-08	0.0297
Our Alg. 3.2	5.72E-08	0.0333	3.31E-10	0.0332	1.28E-08	0.0292

Table 2 Numerical results of our algorithms with different α_n in Example 4.1 ($m = 20$)

Algorithms	Our Algorithm 3.1		Our Algorithm 3.2	
	D_n	CPU (s)	D_n	CPU (s)
$\alpha_n = 1/(n+1)^{0.2}$	1.26E-88	0.0440	2.08E-88	0.0338
$\alpha_n = 1/(n+1)^{0.4}$	1.53E-51	0.0341	5.09E-52	0.0311
$\alpha_n = 1/(n+1)^{0.6}$	1.98E-34	0.0321	8.47E-35	0.0404
$\alpha_n = 1/(n+1)^{0.8}$	8.35E-21	0.0363	1.93E-22	0.0337
$\alpha_n = 1/(n+1)^{1.0}$	8.62E-11	0.0314	5.24E-12	0.0305

Case 4: Compare the parameter α_n. Take $\mu = 0.8$, $\tau_1 = 1$, $\theta = 0.7$, $\epsilon_n = 1/(n+1)^2$, $\beta_n = 0.9(1-\alpha_n)$ and $\xi_n = 1/(n+1)^{1.1}$ for the suggested Algorithms 3.1 and 3.2. The numerical results of our methods with different parameters α_n are shown in Table 2.

Next, we compare the proposed algorithms with Algorithm (MaTEGM) Thong and Vuong (2019) and Algorithm (ViSEGM) Thong et al. (2020). According to the parameters selection of the algorithms presented in the literature (Thong and Vuong 2019; Thong et al. 2020) and the previous analysis of the parameters of our algorithms, the parameters of all algorithms in Example 4.1 are set as follows.

1. Take $\mu = 0.8$, $\tau_1 = 1$, $\theta = 0.7$, $\epsilon_n = 1/(n+1)^2$, $\alpha_n = 1/(n+1)^{0.2}$, $\beta_n = 0.9(1-\alpha_n)$ and $\xi_n = 1/(n+1)^{1.1}$ for the suggested Algorithms 3.1 and 3.2.
2. Pick $\theta = 1$, $\tau_1 = 1$, $\mu = 0.9$, $\epsilon_n = 1/(n+1)^2$, $\alpha_n = 1/(n+1)$ and $f(x) = 0.1x$ for the Algorithm (ViSEGM).
3. Choose $\gamma = 0.1$, $\ell = 0.5$, $\mu = 0.8$, $\alpha_n = 1/(\sqrt{n}+2)$, $\beta_n = 0.5(1-\alpha_n)$ for the Algorithm (MaTEGM).

Fig. 3 shows that our algorithms may perform better in four different dimensions.

The following problem was used by many scholars to test the computational performance of their proposed algorithms for solving pseudomonotone variational inequality problems in infinite-dimensional Hilbert spaces (see, e.g., Hieu et al. (2021); Tan et al. (2021)).

Example 4.2 We consider an example that appears in the infinite-dimensional Hilbert space $\mathcal{H} = L^2[0,1]$ with inner product

$$\langle x, y\rangle = \int_0^1 x(t)y(t)\mathrm{d}t, \ \forall x, y \in \mathcal{H}$$

and induced norm

$$\|x\| = \left(\int_0^1 |x(t)|^2\mathrm{d}t\right)^{1/2}, \ \forall x \in \mathcal{H}.$$

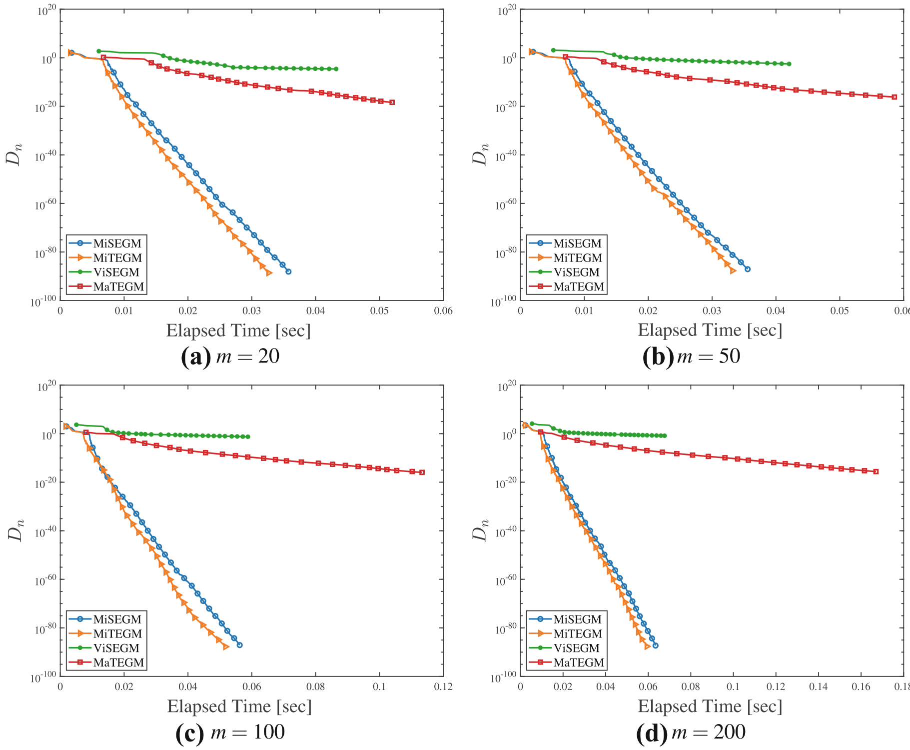


Fig. 3 Numerical results of all algorithms for Example 4.1

Let r, R be two positive real numbers such that $R/(k+1) < r/k < r < R$ for some $k > 1$. Let the feasible set be given by $C = \{x \in \mathcal{H} : \|x\| \leq r\}$ and the operator $A : \mathcal{H} \to \mathcal{H}$ be defined as follows

$$A(x) = (R - \|x\|)x, \quad \forall x \in \mathcal{H}.$$

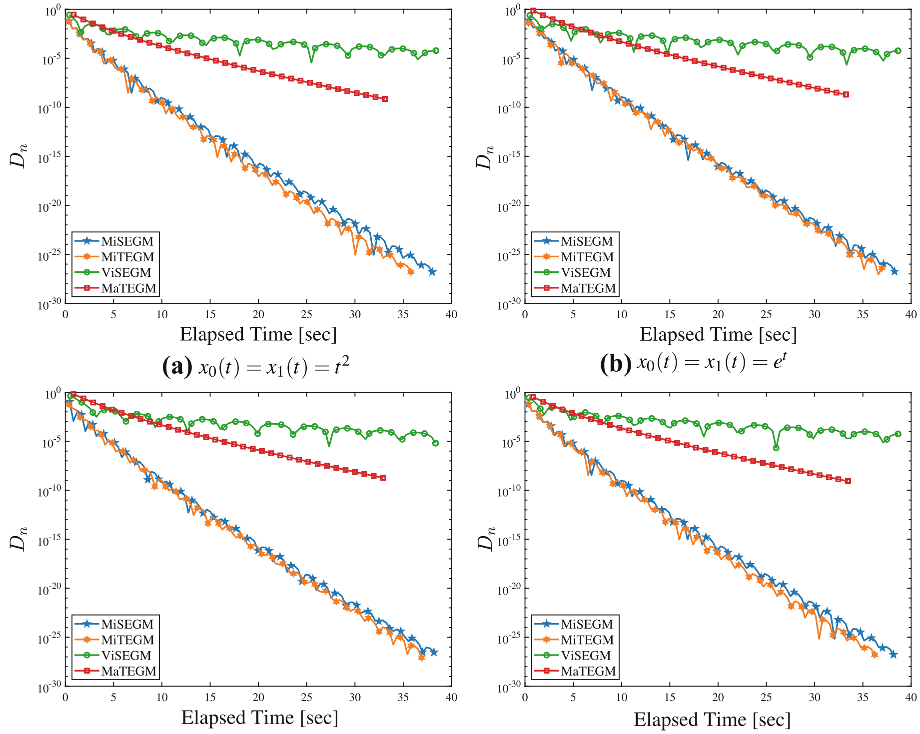
Note that A is not monotone. Indeed, take a particular pair $(\tilde{x}, k\tilde{x})$, we choose $\tilde{x} \in C$ such that $R/(k+1) < \|\tilde{x}\| < r/k$, one can sees that $k\tilde{x} \in C$. By a straightforward computation, we have

$$\langle A(\tilde{x}) - A(\tilde{y}), \tilde{x} - \tilde{y}\rangle = (1-k)^2\|\tilde{x}\|^2(R - (1+k)\|\tilde{x}\|) < 0.$$

Hence, the operator A is not monotone on C. Next we show that A is pseudomonotone. Indeed, if $\langle A(x), y - x\rangle \geq 0$ for all $x, y \in C$, that is, $\langle (R - \|x\|)x, y - x\rangle \geq 0$. Since $\|x\| < R$, we have $\langle x, y - x\rangle \geq 0$. Therefore,

$$\begin{aligned}\langle A(y), y - x\rangle &= \langle (R - \|y\|)y, y - x\rangle \\ &\geq (R - \|y\|)(\langle y, y - x\rangle - \langle x, y - x\rangle) \\ &= (R - \|y\|)\|y - x\|^2 \geq 0.\end{aligned}$$

For the experiment, we choose $R = 1.5$, $r = 1$, $k = 1.1$. The solution of the problem (VIP) with A and C given above is $x^*(t) = 0$. The parameters of all algorithms are set the same as in Example 4.1 and the maximum number of iterations 50 is used as a common stopping



(a) $x_0(t) = x_1(t) = t^2$ **(b)** $x_0(t) = x_1(t) = e^t$ **(c)** $x_0(t) = x_1(t) = \log(t)$ **(d)** $x_0(t) = x_1(t) = \sin(t)$

Fig. 4 Numerical results of all algorithms for Example 4.2

criterion. Figure 4 shows the behaviors of $D_n = \|x_n(t) - x^*(t)\|$ generated by all algorithms with four starting points $x_0 = x_1$. It can be seen from Fig. 4 that the proposed algorithms may perform better.

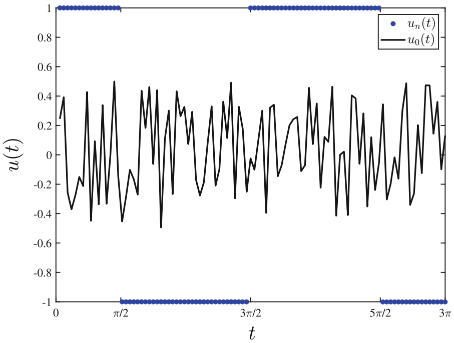
Remark 4.1 Note that the algorithms obtained in this paper can automatically update the step size through a simple calculation, which makes our algorithms work well without the prior information of the Lipschitz constant of the mapping.

4.2 Application to optimal control problems

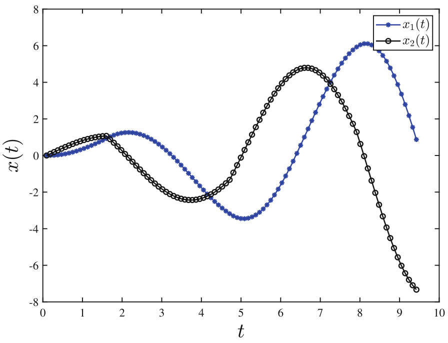
In this subsection, we use the proposed Algorithms 3.1 and 3.2 to solve the (VIP) that appears in optimal control problems. We recommend the reader to refer to Preininger and Vuong (2018); Vuong and Shehu (2019) for the detailed description of the problem.

Example 4.3 (Control of a harmonic oscillator, see Pietrus et al. (2018))

$$\begin{aligned} \text{minimize} \quad & x_2(3\pi) \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = -x_1(t) + u(t), \quad \forall t \in [0, 3\pi], \\ & x(0) = 0, \\ & u(t) \in [-1, 1]. \end{aligned}$$



(a) Initial and optimal controls



(b) Optimal trajectories

Fig. 5 Numerical results of the proposed Algorithm 3.1 for Example 4.3

The exact optimal control of Example 4.3 is known:

$$u^*(t) = \begin{cases} 1, & \text{if } t \in [0, \pi/2) \cup (3\pi/2, 5\pi/2); \\ -1, & \text{if } t \in (\pi/2, 3\pi/2) \cup (5\pi/2, 3\pi]. \end{cases}$$

We compare the proposed algorithms with Algorithm (MaTEGM) Thong and Vuong (2019) and Algorithm (ViSEGM) Thong et al. (2020). Based on the choice of parameters of the methods suggested in Thong and Vuong (2019); Thong et al. (2020) and the analysis of the parameters of our algorithms in Example 4.1, the parameters of all algorithms in Example 4.3 are set as follows.

1. Take $\mu = 0.8$, $\tau_1 = 1$, $\theta = 0.7$, $\epsilon_n = 10^{-4}/(n+1)^2$, $\alpha_n = 1/(n+1)$, $\beta_n = 0.9(1-\alpha_n)$ and $\xi_n = 1/(n+1)^{1.1}$ for the suggested Algorithms 3.1 and 3.2.
2. Pick $\theta = 1$, $\tau_1 = 1$, $\mu = 0.9$, $\epsilon_n = 10^{-4}/(n+1)^2$, $\alpha_n = 10^{-4}/(n+1)$ and $f(x) = 0.1x$ for the Algorithm (ViSEGM).
3. Choose $\gamma = 0.1$, $\ell = 0.5$, $\mu = 0.8$, $\alpha_n = 10^{-4}/(\sqrt{n}+2)$, $\beta_n = 0.5(1-\alpha_n)$ for the Algorithm (MaTEGM).

The initial controls $u_0(t) = u_1(t)$ are randomly generated in $[-1, 1]$, and the stopping criterion is $D_n = \|u_{n+1} - u_n\| \le 10^{-4}$. Figure 5 shows the approximate optimal control and the corresponding trajectories of the proposed Algorithm 3.1.

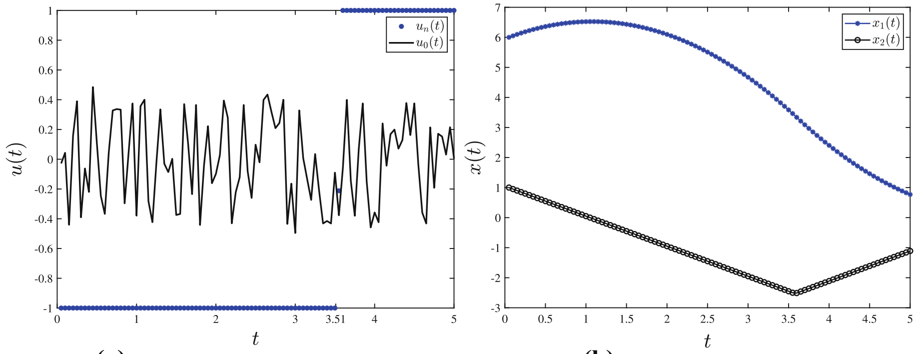
We now consider an example in which the terminal function is not linear.

Example 4.4 (Rocket car (Preininger and Vuong 2018))

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\left((x_1(5))^2 + (x_2(5))^2\right), \\ \text{subject to} \quad & \dot{x}_1(t) = x_2(t), \\ & \dot{x}_2(t) = u(t), \quad \forall t \in [0, 5], \\ & x_1(0) = 6, \quad x_2(0) = 1, \\ & u(t) \in [-1, 1]. \end{aligned}$$

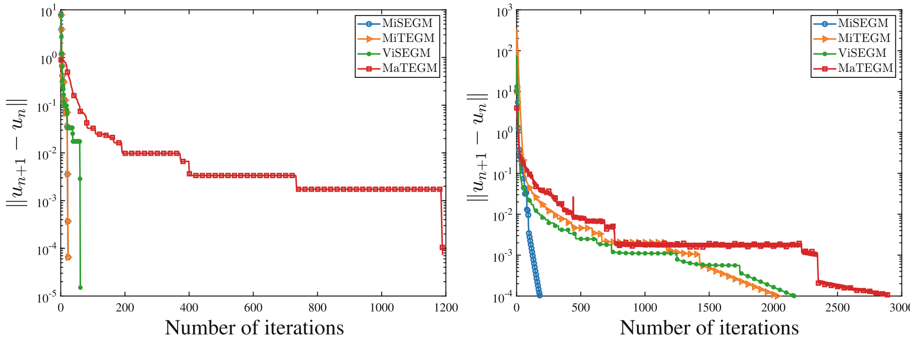
The exact optimal control of Example 4.4 is

$$u^*(t) = \begin{cases} 1 & \text{if } t \in (3.517, 5]; \\ -1 & \text{if } t \in (0, 3.517]. \end{cases}$$



(a) Initial and optimal controls **(b)** Optimal trajectories

Fig. 6 Numerical results of the proposed Algorithm 3.2 for Example 4.4



(a) Example 4.3 **(b)** Example 4.4

Fig. 7 Numerical behaviors of all algorithms for Examples 4.3 and 4.4

Table 3 Numerical results of all algorithms for Examples 4.3 and 4.4

Algorithms	Example 4.3			Example 4.4		
	Iter	CPU (s)	D_n	Iter	CPU (s)	D_n
MiSEGM	22	0.0351	6.4873E-05	180	0.0603	9.9526E-05
MiTEGM	22	0.0205	6.4873E-05	2029	0.6776	9.9787E-05
ViSEGM	60	0.0490	1.4997E-05	2162	0.7564	9.9912E-05
MaTEGM	1189	0.4610	7.2263E-05	2889	1.8460	9.8281E-05

In this example, the parameters of all algorithms are set the same as in Example 4.3. The approximate optimal control and the corresponding trajectories of the suggested Algorithm 3.2 are plotted in Fig. 6.

Finally, the numerical performances of the proposed methods with Algorithm (MaTEGM) Thong and Vuong (2019) and Algorithm (ViSEGM) Thong et al. (2020) in Examples 4.3 and 4.4 are shown in Fig. 7 and Table 3. As in the previous numerical experiments, the proposed algorithms may perform better.

5 Conclusions

In this paper, we proposed two self-adaptive inertial extragradient algorithms to address the variational inequality problem involving L-Lipschitz continuous and pseudomonotone operator but L is unknown. The algorithms are constructed around the inertial method, the Mann-type method, the subgradient extragradient method, and the Tseng's extragradient method. Strong convergence theorems of the proposed methods are established without the prior knowledge of the Lipschitz constant of the operator. Numerical examples show that the proposed algorithms outperform some other relevant methods in the literature. The iterative schemes obtained in this paper improved and extended some previously known results in the field.

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Declarations

Conflict of interest The authors declare that there have no conflict of interest.

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