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An explicit extragradient algorithm for equilibrium problems on Hadamard manifolds

Jingjing Fan[1] · **Bing Tan**[1] · **Songxiao Li**[1]

Abstract

In this paper, we investigate a new extragradient algorithm for solving pseudomonotone equilibrium problems on Hadamard manifolds. Our algorithm uses a variable stepsize, which is updated at each iteration and based on some previous iterates. The convergence analysis of the proposed algorithm is discussed under mild assumptions. In the case where the equilibrium bifunction is strongly pseudomonotone, the R-linear rate of convergence of the new algorithm is formulated. A fundamental experiment is provided to illustrate the numerical behavior of the algorithm.

1 Introduction

Let C be a nonempty convex and closed subset of a real Hilbert space H. Let $f : C \times C \to \mathbb{R}$ be a bifunction with $f(x, x) = 0$ for all $x \in C$. Consider the problem involving f, which consists of finding $x^* \in C$ such that $f(x^*, y) \geq 0, \ \forall y \in C$. The problem, which is also called the Ky Fan inequality, was introduced by Fan (1972) and further developed by Blum and Oettli (1994). It is now known and called the equilibrium problem. The set of all solutions of the equilibrium problem is denoted by $EP(f, C)$. Many problems arising in transportation, financial engineering, and medical imaging can be reduced to finding solutions

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of the equilibrium problems; see, for example, Jadamba et al. (2014), Fan et al. (2020), Tan et al. (2020), Tan et al. (2021) and the references therein.

Recently, many numerical algorithms have been proposed for solving equilibrium problems such as the proximal point algorithm (Chen et al. 2015; Cho 2020), the extragradient algorithm (Vinh and Gibali 2019), the subgradient algorithm (Burachik et al. 2008), and the gap function algorithm (Mastroeni 2003).

In 2019, Hieu et al. (2019) introduced an extragradient algorithm to solve a pseudomonotone equilibrium problem with a Lipschitz-type condition in H. The extragradient algorithm reads as following. Given $x_0 \in C$ and $\lambda_0 > 0$, $\mu \in (0, 1)$, compute y_n and x_{n+1} by

$$\begin{cases} y_n = \arg\min_{y \in C} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} \|x_n - y\|^2 \right\}, \\ x_{n+1} = \arg\min_{y \in C} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} \|x_n - y\|^2 \right\}, \end{cases}$$

where

$$\lambda_{n+1} = \min \left\{ \lambda_n, \frac{\mu(\|x_n - y_n\|^2 + \|x_{n+1} - y_n\|^2)}{2[f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1})]_+} \right\}.$$

They proved that iterative sequence $\{x_n\}$ converges to some $x^* \in EP(f, C)$.

On the other hand, in many practical applications, the natural structure of the data can be modeled as constrained optimization problems, where the constraints are non-linear and non-convex. More specially, the constraints are Riemannian manifolds; see, e.g., Bačák et al. (2016) and Bergmann et al. (2016). Many issues in nonlinear analysis such as fixed point problems, and variational inequalities have been magnified from linear settings to nonlinear systems because these problems cannot be posted in linear spaces and require a manifold structure. Therefore, the extensions of the concepts and techniques in equilibrium problems and related topics from Euclidian spaces to Riemannian manifolds are natural, and the generalizations of optimization methods from Euclidean spaces to Riemannian manifolds also have some more important advantages, see, for example, Li et al. (2019), Dedieu et al. (2003), Li and Wang (2006), Ansari et al. (2020) and Chen et al. (2020).

In 2012, Colao et al. (2012) on the Riemannian setting first introduced the equilibrium problems, which consists of finding $x^* \in C$ such that

$$f\left(x^*, y\right) \geq 0, \quad \forall y \in C, \tag{EP}$$

where C is a nonempty convex and closed subset of a Hadamard manifold $\mathcal{M}$, and $f : C \times C \to \mathbb{R}$ is a bifunction with $f(x, x) = 0$, for all $x \in C$. We denote by $EP(f, C)$ the solution set of problem (EP). Indeed, in recent years, various algorithms, which involves monotone bifunctions, have been extended to solve equilibrium problems from Hilbert spaces to the more general setting of Riemannian manifolds. In particular, Khammahawong et al. (2020) presented an extragradient algorithm to solve strongly pseudomonotone equilibrium problems on Hadamard manifolds. Their algorithm is described as follows. Given $x_0, y_0 \in C$, compute x_{n+1} and y_{n+1} by

$$\begin{cases} x_{n+1} = \arg\min_{y \in C} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \\ y_{n+1} = \arg\min_{y \in C} \left\{ f(y_n, y) + \frac{1}{2\lambda_{n+1}} d^2(x_{n+1}, y) \right\}, \end{cases}$$

where $d(x, y)$ is the geodesic distance between x and y in $\mathcal{M}$ and nonincreasing sequence $\{\lambda_n\}$ satisfying $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=0}^{\infty} \lambda_n = +\infty$. The convergence of sequence $\{x_n\}$ was investigated and obtained.

Inspired by the work in Hieu et al. (2019) and Khammahawong et al. (2020), the aim of this paper is to present an extragradient algorithm with new stepsize rules for pseudomonotone equilibrium problems on Hadamard manifolds and study its convergence properties. Our algorithm uses a variable stepsize sequence, which is generated at each iteration, based on some previous iterates, and without any linesearch procedure. This leads to the main advantage of the algorithm, that is, the performance of its convergence is done without the prior knowledge of the Lipschitz-type constants of bifunctions. The convergence of the resulting algorithm is established under suitable conditions. In the case that the bifunction is strongly pseudomonotone, the R-linear rate of the convergence of the algorithm is also proved.

The rest of this paper is organized as follows. In Sect. 2, we present some basic definitions and fundamental results from manifolds which will be needed in the sequel. In Sect. 3, we propose the new extragradient algorithm involving pseudomonotone bifunctions and analyze its convergence on Hadamard manifolds. In Sect. 4, we study the convergence rate of the proposed algorithm. In Sect. 5, we give numerical experiments to illustrate the computational performance on a test problem. Finally, Sect. 6, the last section, concludes the paper with a brief summary.

2 Preliminaries

In this section, we recall some fundamental definitions, properties, and notations concerned with the Riemannian geometry. These basic facts can be found, for example, in Ferreira et al. (2005), Li et al. (2009) and Ledyaev and Zhu (2007).

Let $\mathcal{M}$ be a finite dimensional differentiable manifold. The set of all tangents at $x \in \mathcal{M}$ is called a tangent space of $\mathcal{M}$ at $x \in \mathcal{M}$, which forms a vector space of the same dimension as $\mathcal{M}$. And we denote it by $T_x\mathcal{M}$. The tangent bundle of $\mathcal{M}$ is denoted by $T\mathcal{M} = \bigcup_{x \in \mathcal{M}} T_x\mathcal{M}$, which is naturally a manifold. We denote by $\langle \cdot, \cdot \rangle_x$ the scalar product on $T_x\mathcal{M}$ with the associated norm $\| \cdot \|_x$, where the subscript x is sometimes omitted. A differentiable manifold $\mathcal{M}$ with a Riemannian metric $\langle \cdot, \cdot \rangle$ is called a Riemannian manifold. Letting $\gamma : [a, b] \to \mathcal{M}$ be a piecewise differentiable curve joining $x = \gamma(a)$ to $y = \gamma(b)$ in $\mathcal{M}$, we can define the length of $L(\gamma) = \int_a^b \|\gamma'(t)\| dt$. The minimal length of all such curves joining x to y is called the Riemannian distance and it is denoted by $d(x, y)$.

Let ∇ be the Levi–Civita connection associated with the Riemannian metric. Let γ be a smooth curve in $\mathcal{M}$. A vector field X is said to be parallel along γ iff $\nabla_{\gamma'} X = 0$. If γ' is parallel along γ, i.e., $\nabla_{\gamma'} \gamma' = 0$, then γ is said to be geodesic. In this case, $\|\gamma'\|$ is a constant. Furthermore, if $\|\gamma'\| = 1$, then γ is called normalized. A geodesic joining x to y in $\mathcal{M}$ is said to be minimal if its length equals $d(x, y)$. Let $\gamma : \mathbb{R} \to \mathcal{M}$ be a geodesic and $P_\gamma[.,.]$ denote the parallel transport along γ with respect to V, which is defined by $P_{\gamma[\gamma(a),\gamma(b)]}(v) = V(\gamma(b))$ for all $a, b \in \mathbb{R}$ and $v \in T_{\gamma(a)}\mathcal{M}$, where V is the unique vector field satisfying $\nabla_{\gamma'(t)} V = 0$ and $V(\gamma(a)) = v$. Then, for any $a, b \in \mathbb{R}$, $P_{\gamma,[\gamma(b),\gamma(a)]}$ is an isometry from $T_{\gamma(a)}\mathcal{M}$ to $T_{\gamma(b)}\mathcal{M}$. We will write $P_{y,x}$ instead of $P_{\gamma,[y,x]}$ in the case where γ is a minimal geodesic joining x to y if this will avoid any confusion.

A Riemannian manifold is complete if, for any $x \in \mathcal{M}$, all geodesics emanating from x are defined for all $-\infty < t < +\infty$. By the Hopf–Rinow Theorem (Sakai 1996), we know that if $\mathcal{M}$ is complete, then any pair of points in $\mathcal{M}$ can be joined by a minimal geodesic. Moreover, $(\mathcal{M}, d)$ is a complete metric space and bounded closed subsets are compact. If $\mathcal{M}$ is a complete Riemannian manifold, then the exponential map $\exp_x : T_x\mathcal{M} \to \mathcal{M}$ at x is defined

by $\exp_x v = \gamma_v(1, x)$ for each $v \in T_x\mathcal{M}$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at x with velocity v, that is, $\gamma(0) = x$ and $\gamma'(0) = v$. It is easy to see that $\exp_x tv = \gamma_v(t, x)$ for each real number t. Note that the mapping $\exp_x$ is differentiable on $T_x\mathcal{M}$ for any $x \in \mathcal{M}$. By the inverse mapping theorem, there exists an inverse exponential map $\exp_x^{-1} : \mathcal{M} \to T_x\mathcal{M}$. Moreover, the geodesic is the unique shortest path with $\| \exp_x^{-1} y \| = \| \exp_y^{-1} x \| = d(x, y)$, where $d(x, y)$ is the geodesic distance between x and y in $\mathcal{M}$. For further details, we refer to Sakai (1996).

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold. If $\mathcal{M}$ is a Hadamard manifold, then $\exp_x^{-1} : \mathcal{M} \to T_x\mathcal{M}$ is a diffeomorphism for every $x \in \mathcal{M}$ and if $x, y \in \mathcal{M}$, then there exists a unique minimal geodesic joining x to y. The rest of the paper, we assume that $\mathcal{M}$ is a Hadamard manifold. The following results are known and will be useful.

Proposition 1 (Sakai 1996) *Let $\mathcal{M}$ be a Hadamard manifold and $p \in \mathcal{M}$. Then, $\exp_p : T_p\mathcal{M} \to \mathcal{M}$ is a diffeomorphism, and for any two points $p, q \in \mathcal{M}$, there exists a unique normalized geodesic joining p to q, which is, in fact, a minimal geodesic.*

This proposition yields that $\mathcal{M}$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Thus, we see that $\mathcal{M}$ has the same topology and differential structure as $\mathbb{R}^n$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties, and one of the most important proprieties is illustrated in the following proposition.

Proposition 2 (Sakai 1996) *Let $\Delta(p_1, p_2, p_3)$ be a geodesic triangle in a Hadamard manifold $\mathcal{M}$. For each $i = 1, 2, 3(\mod 3)$, let $\gamma_i : [0, l_i] \to \mathcal{M}$ denote the geodesic joining p_i to p_{i+1}. Let $l_i = L(\gamma_i)$ and $\alpha_i := \angle(\gamma_i'(0), -\gamma_{i-1}'(l_{i-1}))$ be the angle between tangent vectors $\gamma_i'(0)$ and $\gamma_{i-1}'(l_{i-1})$. Then*

(i) $\alpha_1 + \alpha_2 + \alpha_3 \leq \pi$;
(ii) $l_i^2 + l_{i+1}^2 - 2l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2$;
(iii) $l_{i+1} \cos \alpha_{i+2} + l_i \cos \alpha_i \geq l_{i+2}$.

Considering the distance and the exponential map, we have that the following inequalities are equivalent to Proposition 2 (ii) and (iii).

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\left\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \right\rangle \leq d^2(p_{i-1}, p_i), \quad (1)$$

and

$$d^2(p_i, p_{i+1}) \leq \left\langle \exp_{p_i}^{-1} p_{i+2}, \exp_{p_i}^{-1} p_{i+1} \right\rangle + \left\langle \exp_{p_{i+1}}^{-1} p_{i+2}, \exp_{p_{i+1}}^{-1} p_i \right\rangle,$$

since $\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}$. For further details, we refer to Ferreira and Oliveira (2002).

Lemma 1 (Reich 1980) *Let $\Delta(p, q, r)$ be a geodesic triangle in Hadamard manifold $\mathcal{M}$. Then, there exists a triangle $\Delta(\bar{p}, \bar{q}, \bar{r})$ ($\bar{p}, \bar{q}, \bar{r} \in \mathbb{R}^2$) for $\Delta(p, q, r)$ such that*

$$d(p, q) = \|\bar{p} - \bar{q}\|, \quad d(q, r) = \|\bar{q} - \bar{r}\|, \quad d(r, p) = \|\bar{r} - \bar{p}\|.$$

The triangle $\Delta(\bar{p}, \bar{q}, \bar{r})$ is called the comparison triangle of the geodesic triangle $\Delta(p, q, r)$, which is unique up to isometry of $\mathcal{M}$.

Lemma 2 (Li et al. 2009) *Let $\{x_n\}$ be a sequence in $\mathcal{M}$ such that $x_n \to x_0 \in \mathcal{M}$. Then, the following assertions hold.*

(i) *For any $y \in \mathcal{M}$, we have $\exp_{x_n}^{-1} y \to \exp_{x_0}^{-1} y$ and $\exp_y^{-1} x_n \to \exp_y^{-1} x_0$.*

(ii) *If $v_n \in T_{x_n}\mathcal{M}$ and $v_n \to v_0$, then $v_0 \in T_{x_0}\mathcal{M}$.*

(iii) *Given $u_n, v_n \in T_{x_n}\mathcal{M}$ and $u_0, v_0 \in T_{x_0}\mathcal{M}$, if $u_n \to u_0$ and $v_n \to v_0$, then $\langle u_n, v_n \rangle \to \langle u_0, v_0 \rangle$.*

(iv) *For any $u \in T_{x_0}\mathcal{M}$, the function $A : \mathcal{M} \to T\mathcal{M}$ defined by $A(x) = P_{x,x_0}u$ for each $x \in \mathcal{M}$ is continuous on $\mathcal{M}$.*

Definition 1 A subset C is said to be convex if, for every two points x and y in C, the geodesic joining x to y is contained in C, that is, if $\gamma : [a, b] \to \mathcal{M}$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1 - t)a + tb) \in C$ for all $t \in [0, 1]$.

Definition 2 A real function f defined in $\mathcal{M}$ is said to be convex if, for any geodesic γ of $\mathcal{M}$, the composition function $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1 - t)b) \leq t(f \circ \gamma)(a) + (1 - t)(f \circ \gamma)(b),$$

where $a, b \in \mathbb{R}$, and $t \in [0, 1]$.

Definition 3 Let $f : \mathcal{M} \to \mathbb{R}$ be a convex and $x \in \mathcal{M}$. A vector $p \in T_x\mathcal{M}$ is said to be a subgradient of f at x if, for any $y \in \mathcal{M}$,

$$f(y) \geq f(x) + \langle p, \exp_x^{-1} y \rangle.$$

The set of all subgradients of f, denoted by $\partial f(x)$, is called the subdifferential of f at x, which is closed convex set. Let $D(\partial f)$ denote the domain of ∂f defined by $D(\partial f) = \{x \in \mathcal{M} : \partial f(x) \neq \emptyset\}$. The existence of subgradients for convex functions is guaranteed by the following proposition.

Proposition 3 (Ferreira and Oliveira 2002) *Let $\mathcal{M}$ be a Hadamard manifold and $f : [a, b] \to \mathbb{R}$ be convex. Then, for all $x \in \mathcal{M}$, the subdifferential $\partial f(x)$ of f at x is nonempty. That is, $D(\partial f) = \mathcal{M}$.*

Next, we recall some concepts of monotonicity of a bifunction.

Definition 4 (Németh 1998, 1999) A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(i) monotone if, for any $(x, y) \in C \times C$,

$$f(x, y) + f(y, x) \leq 0;$$

(ii) strongly monotone if, for any $(x, y) \in C \times C$, there exists a positive constant γ such that

$$f(x, y) + f(y, x) \leq -\gamma d^2(x, y);$$

(iii) pseudomonotone if, for any $(x, y) \in C \times C$,

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0;$$

(iv) strongly pseudomonotone if, for any $(x, y) \in C \times C$, there exists a positive constant γ such that

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq -\gamma d^2(x, y).$$

It follows from the definitions that the following implications hold:

$$(ii) \Rightarrow (i) \Rightarrow (iii) \quad \text{and} \quad (ii) \Rightarrow (iv) \Rightarrow (iii).$$

Definition 5 (Mastroeni [2003]) A bifunction $f : C \times C \to \mathbb{R}$ is said to satisfy a Lipschitz-type condition on C if there exist two positive constants γ_1 and γ_2 such that

$$f(x, y) + f(y, z) \geq f(x, z) - \gamma_1 d^2(x, y) - \gamma_2 d^2(y, z), \quad \forall x, y, z \in C.$$

Let $f : \mathcal{M} \to \mathbb{R}$ be a convex, proper and lower semicontinuous function. The proximal point algorithm generates, for a initial point $x_0 \in \mathcal{M}$, a sequence $\{x_n\} \subset \mathcal{M}$, which is defined by the following:

$$x_{n+1} = \arg\min_{t \in \mathcal{M}} \left\{ f(t) + \frac{\lambda_n}{2} d^2(x_n, t) \right\}, \quad \lambda_n \subset (0, +\infty). \tag{2}$$

The following lemmas are useful for the convergence of our proposed algorithm.

Lemma 3 (Ferreira and Oliveira [2002]) *Let $f : \mathcal{M} \to \mathbb{R}$ be a convex, proper and lower semicontinuous function. Then, the sequence $\{x_n\}$ generated by* (2) *is well defined, and characterized by*

$$\lambda_n(\exp_{x_{n+1}}^{-1} x_n) \in \partial f(x_{n+1}).$$

Definition 6 (Ferreira and Oliveira [2002]) Let X be a complete metric space and let $C \in X$ be a nonempty set. A sequence $\{x_n\} \subset X$ is said to be Fej*é*r convergent to C if, for all $y \in C$ and $n \geq 0$, $d(x_{n+1}, y) \leq d(x_n, y)$.

Lemma 4 (Li et al. [2009]) *Let X be a complete metric space and let $C \in X$ be a nonempty set. Let $\{x_n\} \subset X$ be Fejér convergent to C and suppose that any cluster point of $\{x_n\}$ belongs to C. Then $\{x_n\}$ converges to some point in C.*

3 The explicit extragradient algorithm for the equilibrium problem

In this section, we introduce an extragradient algorithm for the equilibrium problem (EP) involving a pseudomonotone bifunction on Hadamard manifolds. Unlike existing extragradient-like methods for problem (EP), the stepsizes used in the presented algorithm are independent of the Lipschitz-type constants. From now on, let C be a nonempty closed convex set of $\mathcal{M}$ and let $f : C \times C \to \mathbb{R}$ be a bifunction. To obtain the convergence of Algorithm 1, we make the following hypothesizes regarding the bifunction:

(C1) f is pseudomontone on C and $f(x, x) = 0$ for all $x \in C$;
(C2) f satisfies the Lipschitz-type condition;
(C3) $f(x, \cdot)$ is convex and lower semicontinuous on C for all $x \in C$;
(C4) $\limsup_{n \to \infty} f(x_n, y) \leq f(x, y)$ for each $y \in C$.

For the sake of simplicity in the presentation, we will use the notation $[t]_+ = \max\{0, t\}$ and adopt the conventions $\frac{0}{0} = +\infty$ and $\frac{a}{0} = +\infty$ ($a \neq 0$). More precisely, the algorithm is described as follows:

Remark 1 Under hypothesis (C2), we see that there exist constants $\gamma_1 > 0$, $\gamma_2 > 0$ such that

$$f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq \gamma_1 d^2(x_n, y_n) + \gamma_2 d^2(x_{n+1}, y_n)$$
$$\leq \max\{\gamma_1, \gamma_2\}(d^2(x_n, y_n) + d^2(x_{n+1}, y_n)).$$

Thus, from the definition of $\{\lambda_n\}$, we see that this sequence is bounded from below by $\left\{\lambda_0, \frac{\mu}{2\max\{\gamma_1, \gamma_2\}}\right\}$. Moreover, $\{\lambda_n\}$ is non-increasing monotone. Therefore, there exists a real

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Algorithm 1 (An explicit extragradient algorithm for pseudomonotone EPs)

Initialization: Choose $x_0 \in C$ and $\lambda_0 > 0$, $\mu \in (0, 1)$.

Iterative Steps: Given the current iterate $x_n \in C$ and $\lambda_n (n \geq 0)$, calculate x_{n+1}, λ_{n+1} as follows. Compute

$$\begin{cases} y_n = \arg\min_{y \in C} \left\{ f(x_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \\ x_{n+1} = \arg\min_{y \in C} \left\{ f(y_n, y) + \frac{1}{2\lambda_n} d^2(x_n, y) \right\}, \end{cases}$$

and set

$$\lambda_{n+1} = \min \left\{ \lambda_n, \frac{\mu(d^2(x_n, y_n) + d^2(x_{n+1}, y_n))}{2[f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1})]_+} \right\}.$$

Stopping Criterion: If $y_n = x_n$, then stop and x_n is the solution of equilibrium problem (EP).

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number $\lambda > 0$ such that $\lim_{n \to \infty} \lambda_n = \lambda$. If $f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq 0$, then we see from the definition of $\{\lambda_{n+1}\}$ that $\lambda_{n+1} := \lambda_n$.

We are now turn to the main result regarding the convergence of the proposed algorithm.

Theorem 1 *Assume that bifunction f satisfies (C1)–(C4). Then, the sequences $\{x_n\}$ generated by Algorithm 1 converges to a solution of equilibrium problem (EP).*

Proof From the definition of x_{n+1} and Lemma 3, one obtains

$$\left\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \right\rangle \leq \lambda_n f(y_n, y) - \lambda_n f(y_n, x_{n+1}), \quad \forall y \in C. \tag{3}$$

From the definition of λ_{n+1}, one concludes

$$f(x_n, x_{n+1}) - f(x_n, y_n) - f(y_n, x_{n+1}) \leq \frac{\mu(d^2(x_n, y_n) + d^2(x_{n+1}, y_n))}{2\lambda_{n+1}}.$$

Since $\lambda_n > 0$, one can write the above inequality as

$$\lambda_n f(y_n, x_{n+1}) \geq \lambda_n (f(x_n, x_{n+1}) - f(x_n, y_n)) - \frac{\mu \lambda_n (d^2(x_n, y_n) + d^2(x_{n+1}, y_n))}{2\lambda_{n+1}}. \tag{4}$$

Combining (4) through (3) yields that

$$-\left\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \right\rangle \geq \lambda_n (f(x_n, x_{n+1}) - f(x_n, y_n)) - \frac{\mu \lambda_n}{2\lambda_{n+1}} d^2(x_n, y_n) \\ - \frac{\mu \lambda_n}{2\lambda_{n+1}} d^2(x_{n+1}, y_n) - \lambda_n f(y_n, y). \tag{5}$$

It also follows from the definition of y_n and Lemma 3 that

$$\lambda_n (f(x_n, x_{n+1}) - f(x_n, y_n)) \geq \left\langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \right\rangle. \tag{6}$$

From relations (5) and (6), one obtains

$$-2\left\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \right\rangle \geq 2\left\langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \right\rangle - \frac{\mu \lambda_n}{\lambda_{n+1}} d^2(x_n, y_n) \\ - \frac{\mu \lambda_n}{\lambda_{n+1}} d^2(x_{n+1}, y_n) - 2\lambda_n f(y_n, y). \tag{7}$$

Let $\Delta(x_{n+1}, x_n, y) \subseteq \mathcal{M}$ be the geodesic triangle. Using (1), one concludes that

$$2\left\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y \right\rangle \geq d^2(x_n, x_{n+1}) + d^2(x_{n+1}, y) - d^2(x_n, y). \tag{8}$$

Similarly, let $\Delta(x_n, x_{n+1}, y_n) \subseteq \mathcal{M}$ be the geodesic triangle. It follows from (1) that

$$2\left\langle \exp_{y_n}^{-1} x_n, \exp_{y_n}^{-1} x_{n+1} \right\rangle \geq d^2(y_n, x_n) + d^2(y_n, x_{n+1}) - d^2(x_n, x_{n+1}). \tag{9}$$

Combining (7)–(9), one arrives at

$$\begin{aligned}
d^2(x_{n+1}, y) \leq{} & d^2(x_n, y) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n) \\
& - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) + 2\lambda_n f(y_n, y).
\end{aligned} \tag{10}$$

Taking $p \in EP(f, C)$, we have that $f(p, y_n) \geq 0$. It follows from the pseudomonotonicity of f that $f(y_n, p) \leq 0$. Then, using $y = p \in C$ in (10), we get

$$\begin{aligned}
d^2(x_{n+1}, p) \leq{} & d^2(x_n, p) - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n) \\
& - \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n).
\end{aligned} \tag{11}$$

Let κ be fixed in $(0, 1 - \mu)$. Since $\lim_{n\to\infty} \lambda_n = \lambda > 0$, one asserts that

$$\lim_{n\to\infty} \left(1 - \frac{\mu \lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > \kappa > 0.$$

Thus, there exists $n_0 \geq 0$ such that, for all $n \geq n_0$,

$$1 - \frac{\mu \lambda_n}{\lambda_{n+1}} > \kappa > 0. \tag{12}$$

Adding (12) into (11), one obtains

$$d^2(x_{n+1}, p) \leq d^2(x_n, p) - \kappa(d^2(x_n, y_n) + d^2(x_{n+1}, y_n)),$$

which implies that

$$a_{n+1} \leq a_n - b_n, \tag{13}$$

where

$$a_n = d^2(x_n, p) \text{ and } b_n = \kappa(d^2(x_n, y_n) + d^2(x_{n+1}, y_n)).$$

It is obvious that $\lim_{n\to\infty} a_n$ exists, and $\lim_{n\to\infty} b_n = 0$. Hence $\{x_n\}$ is bounded. Thus, we conclude from the definition of b_n that

$$\lim_{n\to\infty} d^2(x_n, y_n) = \lim_{n\to\infty} d^2(x_{n+1}, y_n) = 0, \tag{14}$$

which implies from the boundedness of $\{x_n\}$ that $\{y_n\}$ is bounded too. Using (1), we obtain

$$d^2(x_n, x_{n+1}) \leq 2\left\langle \exp_{x_{n+1}}^{-1} x_n, \exp_{x_{n+1}}^{-1} y_n \right\rangle - d^2(x_{n+1}, y_n) + d^2(x_n, y_n).$$

We also have

$$\lim_{n\to\infty} d^2(x_n, x_{n+1}) = 0. \tag{15}$$

We next prove that each cluster point of $\{x_n\}$ is in $EP(f, C)$. We show that $\{x_n\}$ is bounded. Therefore there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $x^* \in C$ such that x^* is a cluster point of $\{x_n\}$, i.e., $x_{n_k} \to x^*$. Hence, using (14), we have that $y_{n_k} \to x^*$. Replacing n by n_k in (10), taking lim sup and using hypothesis (C4), we have

$$f(x^*, y) \geq \limsup_{k \to \infty} f(y_{n_k}, y) \geq \frac{1}{2\lambda} \limsup_{k \to \infty} (d^2(x_{n_k+1}, y) - d^2(x_{n_k}, y)), \quad \forall y \in C. \quad (16)$$

On the other hand, from (1), we obtain

$$d^2(x_{n_k+1}, y) - d^2(x_{n_k}, y) \leq \left\langle \exp^{-1}_{x_{n_k+1}} x_{n_k}, \exp^{-1}_{x_{n_k+1}} y \right\rangle - d^2(x_{n_k+1}, x_{n_k}).$$

This together with (15) implies that

$$\lim_{k \to \infty} (d^2(x_{n_k+1}, y) - d^2(x_{n_k}, y)) = 0. \quad (17)$$

Combining (16) and (17), we get $f(x^*, y) \geq \limsup_{k \to \infty} f(y_{n_k}, y) \geq 0$, for all $y \in C$. Therefore, $x^* \in EP(f, C)$. From (11), (12) and Definition 6, we know that $\{x_n\}$ is Fejér convergent to C. Finally, Lemma 4 implies that sequence $\{x_n\}$ converges to a point of $EP(f, C)$. This completes the proof. □

4 The R-linear rate of the convergence

The algorithms in Khammahawong et al. (2020) have some special advantages that they are done without the prior knowledge of the Lipschitz-type constants of the bifunction. However, in the case that f is strongly pseudomonotone (SP), the linear rate of convergence cannot be obtained for these algorithms. In this section, we will establish the R-linear rate of the convergence of Algorithms 1 under hypothesis (SP) and (C1)–(C4). Under these assumptions, equilibrium problem (EP) has a unique solution, denoted by $\bar{x}$. The rate of the convergence of the proposed algorithm is ensured by the following theorem.

Theorem 2 *Under hypotheses (C1)–(C4) and (SP), the sequence $\{x_n\}$ generated by Algorithm 1 converges R-linearly to the unique solution $\bar{x}$ of equilibrium problem (EP).*

Proof Using (10) with $y = \bar{x}$, we obtain

$$d^2(x_{n+1}, \bar{x}) \leq d^2(x_n, \bar{x}) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n)$$
$$- \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) + 2\lambda_n f(y_n, \bar{x}). \quad (18)$$

Since $\bar{x} \in EP(f, C)$ we have $f(\bar{x}, y_n) \geq 0$. From assumption (SP), we get that

$$f(y_n, \bar{x}) \leq -\rho d^2(y_n, \bar{x}), \quad (19)$$

where ρ is a positive real number. Adding (19) into (18), we have

$$d^2(x_{n+1}, \bar{x}) \leq d^2(x_n, \bar{x}) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n)$$
$$- \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) - 2\rho\lambda_n d^2(y_n, \bar{x}). \quad (20)$$

Since $\{\lambda_n\}$ is non-increasing monotone and $\lim_{n\to\infty} \lambda_n = \lambda > 0$, one has that $\lambda_n \geq \lambda_\infty = \lambda$ for all $n \geq 0$. Then, it follows from (20) that

$$
\begin{aligned}
d^2(x_{n+1}, \bar{x}) \leq & d^2(x_n, \bar{x}) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n) \\
& - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_{n+1}, y_n) - 2\rho\lambda d^2(y_n, \bar{x}).
\end{aligned}
\tag{21}
$$

Letting κ be fixed in $(0, \frac{1-\mu}{2})$, we find that

$$
\lim_{n\to\infty} \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) = 1 - \mu > 2\kappa > 0.
$$

Thus, there exists $n_0 \geq 0$ such that, for all $n \geq n_0$,

$$
1 - \frac{\mu\lambda_n}{\lambda_{n+1}} > 2\kappa > 0.
\tag{22}
$$

It follows from (21) and (22) that, for all $n \geq n_0$,

$$
\begin{aligned}
d^2(x_{n+1}, \bar{x}) &\leq d^2(x_n, \bar{x}) - \left(1 - \frac{\mu\lambda_n}{\lambda_{n+1}}\right) d^2(x_n, y_n) - 2\rho\lambda d^2(y_n, \bar{x}) \\
&\leq d^2(x_n, \bar{x}) - 2\kappa d^2(x_n, y_n) - 2\rho\lambda d^2(y_n, \bar{x}) \\
&\leq d^2(x_n, \bar{x}) - \min\{\kappa, \rho\lambda\} \left\{2d^2(x_n, y_n) + 2d^2(y_n, \bar{x})\right\} \\
&\leq d^2(x_n, \bar{x}) - \min\{\kappa, \rho\lambda\} d^2(x_n, \bar{x}) \\
&= r d^2(x_n, \bar{x}),
\end{aligned}
\tag{23}
$$

where $r = 1 - \min\{\kappa, \rho\lambda\} \in (0, 1)$. In view of (23), one concludes that

$$
d^2(x_{n+1}, \bar{x}) \leq r^{n-n_0+1} d^2(x_{n_0}, \bar{x}), \quad \forall n \geq n_0,
$$

or $d^2(x_{n+1}, \bar{x}) \leq Mr^n$ for all $n \geq n_0$, where $M = r^{1-n_0} d^2(x_{n_0}, \bar{x})$. This finishes the proof. $\qquad\square$

5 Numerical experiment

In this section, we illustrate the convergence behavior of our proposed Algorithm 1 through a numerical experiment, which is relative to a strongly pseudomonotone bifunction. We use the *fmincon* function in the MATLAB Optimization toolbox to solve the optimization problem. All the programs are executed in MATLAB2018a on a PC Desktop Intel(R) Core(TM) i5-8250U CPU @ 1.60GHz 1.800 GHz, RAM 8.00 GB.

Example 1 Form (Ansari et al. 2019, Example 1), let $\mathbb{R}^{++} = \{x \in \mathbb{R} : x > 0\}$ and $\mathcal{M} = \left(\mathbb{R}^{++}, \langle\cdot, \cdot\rangle\right)$ be the Riemanian manifold with the metric $\langle m, n\rangle := mn$. Thus, the sectional curvature of $\mathcal{M}$ is 0. $T_x\mathcal{M}$ denotes the tangent space at $x \in \mathcal{M}$, equals $\mathbb{R}$. The Riemannian distance $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}^+$ is defined by $d(x, y) := |\ln(x/y)|$. Then $\mathcal{M}$ is a Hadamard manifold. Let $\gamma : [0, 1] \to \mathcal{M}$ be a geodesic starting from $x = \gamma(0)$ with velocity $v = \gamma'(0) \in T_x\mathcal{M}$ defined by $\gamma(t) := xe^{(v/x)t}$. Hence, we get that $\exp_x tv = xe^{(v/x)t}$. For any $x, y \in \mathcal{M}$, we obtain

$$
y = \exp_x \left(d(x, y) \frac{\exp_x^{-1} y}{d(x, y)}\right) = xe^{\left(\frac{\exp_x^{-1} y}{xd(x,y)}\right) d(x,y)} = xe^{\frac{\exp_x^{-1} y}{x}},
$$

and thus, the inverse of exponential map is $\exp_x^{-1} y = x \ln(y/x)$.

Next, we consider an extension of a Nash-Cournot oligopolistic equilibrium model (Facchinei and Pang 2007) with the price function and fee-fax function being affine. Assume that there are n companies. Let $x = (x_1, x_2, \ldots, x_n)$ be a vector, and its elements x_i represent the number of goods produced by company i. We suppose that the price function $p_i(s)$ is a decreasing affine of $s = \sum_{i=1}^n x_i$ such as $p_i(s) = a_i - b_i s$, where $a_i, b_i \geq 0$. Then, the profit of the company i is given by $f_i(x) = p_i(s)x_i - c_i(x_i)$, where $c_i(x_i)$ is the tax and fee for generating x_i. Set $\Phi_i = [x_{i,\min}, x_{i,\max}]$ is the strategy set of the company i. Therefore, $\Phi = \Phi_1 \times \cdots \times \Phi_n$ is the strategy set of the model. In fact, each company i tries to maximize its own profits by choosing the corresponding production level x_i. The common method of this model is based on the well-known Nash equilibrium concept.

We recall that a point $x^* = (x_1^*, x_2^*, \ldots, x_n^*) \in \Phi = \Phi_1 \times \cdots \times \Phi_n$ is called an equilibrium point of the model if

$$f_i(x^*) \geq f_i(x^*[x_i]), \quad \forall x_i \in \Phi_i, \forall i = 1, \ldots, n,$$

where $x^*[x_i]$ stands for the vector obtained from x^* by replacing x_i^* with x_i. Set $f(x, y) = \phi(x, y) - \phi(x, x)$, where $\phi(x, y) = -\sum_{i=1}^n f_i(x[y_i])$. The problem of finding a Nash equilibrium point of the model can be expressed as:

$$\text{Find } x^* \in \Phi, \quad \text{such that } f(x^*, x) \geq 0, \quad \forall x \in \Phi.$$

We suppose that the tax-fee function $c_i(x_i)$ is increasing and affine for every i. This assumption means that as the number of products increases, the taxes and expenses for producing a unit increase. Here, the bifunction f can be expressed as $f(x, y) = \langle Cx + Dy + q, y - x \rangle$, where $q \in \mathbb{R}^n$ and C, D are two matrices of order n such that D is symmetric positive semidefinite and $D - C$ is symmetric negative semidefinite. We consider here that $D - C$ is symmetric negative definite. From the property of $D - C$, if $f(x, y) \geq 0$, we have

$$\begin{aligned}
f(y, x) &\leq f(y, x) + f(x, y) \\
&= \langle Cy + Dx + q, x - y \rangle + \langle Cx + Dy + q, y - x \rangle \\
&= \langle (C - D)y + (D - C)x, x - y \rangle \\
&= (x - y)^{\mathrm{T}}(D - C)(x - y) \\
&\leq -\delta d^2(x, y),
\end{aligned}$$

where $\delta > 0$. Then, f is strongly pseudomonotone, i.e., assumption (C1) holds for f. Furthermore, it is easy to prove that f satisfies the Lipschitz-type condition, see, e.g., Hieu (2016), (C2) is fulfilled. Assumption (C3) and (C4) are automatically fulfilled. Hence, Algorithm 1 can be applied in this case.

For the numerical experiment, we consider four companies, that are defined in Table 1.

Table 1 Parameter settings for Example 1

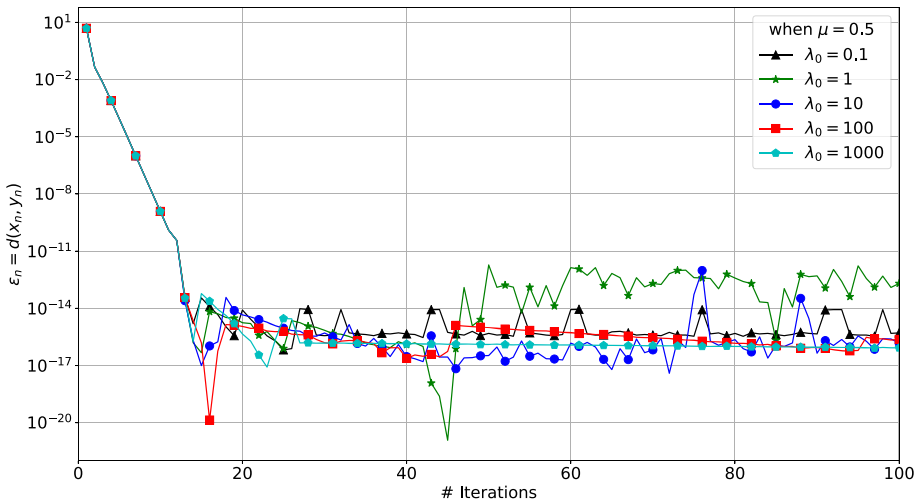| Company i	Price $p_i(s)$	Tax $c_i(x_i)$	Strategy set Φ_i
1	$p_1(s) = 100 - 0.01s$	$c_1(x_1) = 20x_1$	$\Phi_1 = [1000, 2000]$
2	$p_2(s) = 110 - 0.02s$	$c_2(x_2) = 15x_2 + 100$	$\Phi_2 = [500, 2500]$
3	$p_3(s) = 100 - 0.015s$	$c_3(x_3) = 17x_3$	$\Phi_3 = [800, 1500]$
4	$p_4(s) = 115 - 0.05s$	$c_4(x_4) = 20x_4 + 75$	$\Phi_4 = [500, 3000]$

Fig. 1 Numerical behavior of $\{\varepsilon_n\}$ in Algorithm 1 with the number of iterations
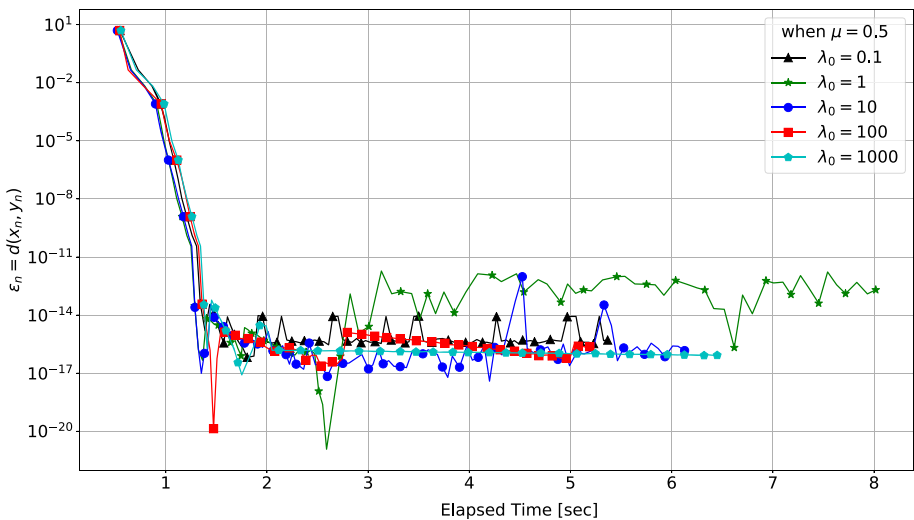

Fig. 2 Numerical behavior of $\{\varepsilon_n\}$ in Algorithm 1 with elapsed time

In our Algorithm 1, The starting point is $x_0 = (1000, 500, 800, 500)^{\mathrm{T}} \in \mathbb{R}^4$. In view of Algorithm 1, we see that $y_n = x_n$, then x_n is the solution of problem (EP). Therefore, we use the sequence $\varepsilon_n = d(x_n, y_n)$ to study the convergence of the Algorithm 1. The convergence of $\{\varepsilon_n\}$ to zero implies that sequence $\{x_n\}$ converges to the solution of the problem. Next, we show the behavior of $\{\varepsilon_n\}$ in Algorithm 1 for different initial λ_0 and μ. Parameter settings for Example 1 in Table 1 (Khammahawong et al. 2020). We perform experiments for both number of iterations (# iteration) and elapsed execution time (Elapsed time [sec]). The numerical results are reported in Figs. 1, 2, 3 and 4. In these figures, the x-axis represents the number of iterations or execution time, and the y-axis represents the value of $\{\varepsilon_n\}$.
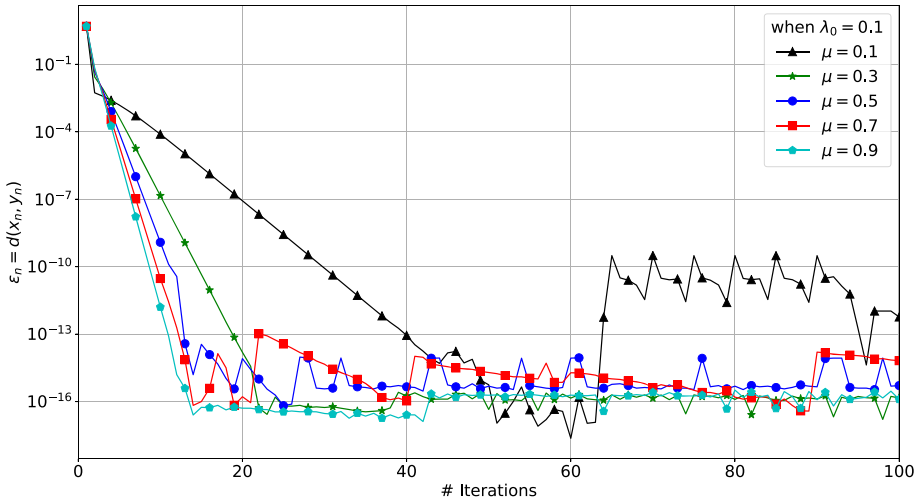
Fig. 3 Numerical behavior of $\{\varepsilon_n\}$ in Algorithm 1 with the number of iterations
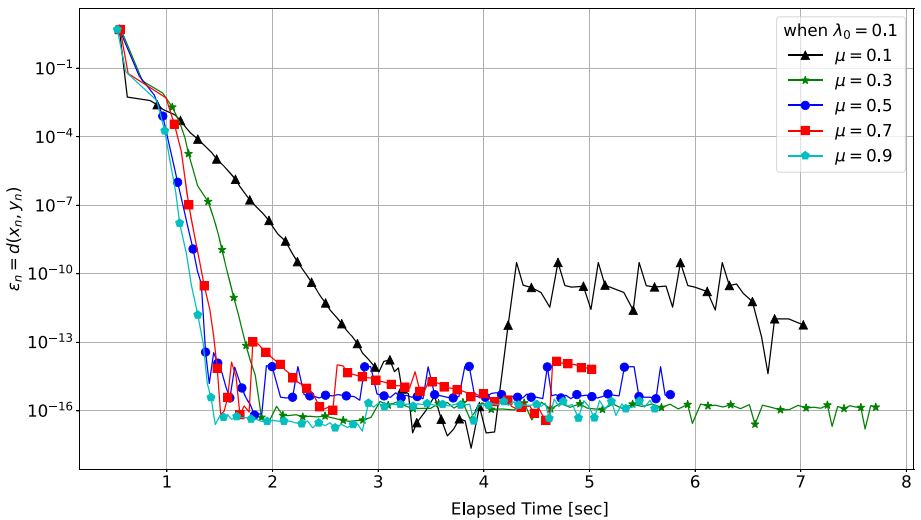

Fig. 4 Numerical behavior of $\{\varepsilon_n\}$ in Algorithm 1 with elapsed time

From Figs. 1, 2, 3 and 4, we know that the rate of convergence of the sequence $\{\varepsilon_n\}$ generated by Algorithm 1 is independent of parameters λ_0 and μ. In addition, the first 20 iterations of $\{\varepsilon_n\}$ are very fast, as the number of iterations increases, it seems to become unstable.

6 Concluding remarks

In this paper, we investigated the convergence of the new extragradient algorithm for the equilibrium problem involving pseudomonotone and Lipschitz-type bifunctions on Hadamard

manifolds. A new stepsize rule allows us not to previously know the information of the Lipschitz-type constants of bifunctions. The convergence as well as the R-linear rate of convergence of the algorithm were constructed. The numerical behaviour of the extragradient algorithm was also discussed. To devise more effective algorithms for problem (EP) on Hadamard manifolds, we will consider the geometric structure of manifolds in the future. It is of interest to do some numerical experiments and comparison with other algorithms for practical problems on Riemannian manifolds.

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